

Twisted Milnor hypersurfaces

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1 Twisted Milnor hypersurfaces

2 Background

3 Our Results

4 Other results

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The twisted Milnor hypersurface $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$

$$H := H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2) \xleftrightarrow{P.D} d_1u + d_2v, \text{ where } \mathbf{I} = (i_1, \dots, i_{n_2}).$$

- $\mathbf{I} = (0, \dots, 0)$: $H_{n_1, n_2}^{\mathbf{0}}(d_1, d_2) \subset V = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$
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- $\mathbf{I} = (1, 0, \dots, 0)$, $V = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2}) = L(n_1, n_2)$ [Lü and Panov,2016]
Any generator in Ω^U can always be found in $\mathbb{Z}\langle L(n_1, n_2) \rangle$.

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$\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \rightarrow \Delta^{n_1} \times \Delta^{n_2}$ as a Quasitoric manifold with characteristic matrix:([Block diagonal matrix](#))

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & \cdots & & -1 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & & & 1 & -1 & 0 \\ & & & & 0 & 1 \\ & & & & & -1 \\ \vdots & & & & \ddots & \cdots \\ 0 & & & & & 1 & -1 \end{bmatrix}$$

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Theorem (Kato, Matsumoto, 1972)

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$$\pi_1(H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)) \xrightarrow{i_*} \pi_1(V) = 0 \quad (\Rightarrow H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2) \text{ is simply connected}).$$



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- $c_1(V) = (n_1 + 1 - \sigma_1)u + (n_2 + 1)v$. ($\sigma_1 = \sum_{j=1}^{n_2} i_j$.) Based on:

Theorem (Borel and Hirzebruch ,1958)

$p : \mathbb{C}P(\xi) \rightarrow X$, let γ be the tautological line bundle over $\mathbb{C}P(\xi)$. Then
 $T\mathbb{C}P(\xi) \oplus \underline{\mathbb{C}} \cong p^*TX \oplus (\bar{\gamma} \otimes p^*\xi)$, $H^*(\mathbb{C}P(\xi); \mathbb{Z}) \cong H^*(X)[v]/c_n(\bar{\gamma} \otimes p^*\xi)$,
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If there exists $k_1, k_2 \in \mathbb{Z}$ such that

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Convention Throughout this talk, $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$ spin means $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$ carries the induced spin structure from the spin^c structure of V . (V is a toric variety and of course complex manifold)

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- * However, $\dim_{\mathbb{R}} H^{\mathbf{I}}_{n_1, n_2}(d_1, d_2) = 2(n_1 + n_2 - 1)$ even,
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He established the analytic Rokhlin congruence formula.

Theorem (Zhang, 1993. The Rokhlin congruence formula)

$$\alpha(H) \equiv \langle \widehat{A}(V) \exp^{\frac{\omega}{2}}, [V] \rangle \pmod{2}.$$

Where $H \xleftrightarrow{P.D} \omega \in H^2(V; \mathbb{Z})$, and $\dim H \equiv 2 \pmod{8}$.

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For any complete intersection $V_{d_1, \dots, d_r}^n \subset \mathbb{C}P^{n+r}$, when $n = 4k + 1$ (i.e. $\dim_{\mathbb{R}} V_{d_1, \dots, d_r}^n \equiv 2 \pmod{8}$),

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- ♣ [Baraglia, 2020] recovered the α -invariant of any **complete intersections** (which obtained in [Fang and Shao, 2009]) with a different approach.

* Let $F_{n_1, n_2, \mathbf{l}}(d_1, d_2) =$

$$\sum_{\substack{0 \leq r \leq n_2 \\ \forall 1 \leq j \leq r \\ l_j \geq 1, l \leq n_1, 0 \leq m_j \leq l_j \\ 1 \leq s_1 < s_2 < \dots < s_r \leq n_2}} (-1)^m \binom{\vec{l}}{\vec{m}} \binom{n_1 + k_1 - \vec{m} \cdot \vec{l}}{n_1} \binom{n_2 + k_2 + l - r}{n_2 + l},$$

where $l = \sum_{j=1}^r l_j$, $m = \sum_{j=1}^r m_j$, $\binom{\vec{l}}{\vec{m}} = \binom{l_1}{m_1} \cdots \binom{l_r}{m_r}$, $\vec{m} \cdot \vec{l} = \sum_{j=1}^r m_j i_{s_j}$.

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$$\begin{aligned} & l_j \geq 1, l \leq n_1, 0 \leq m_j \leq l_j \\ & 1 \leq s_1 < s_2 < \dots < s_r \leq n_2 \end{aligned}$$

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$$d_1 = 2k_1 + n_1 + 1 - \sigma_1, d_2 = 2k_2 + n_2 + 1.$$

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α -invariant

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Theorem

If $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$ is spin with $\dim \geq 5$ and $n_1 + n_2 \equiv 2 \pmod{4}$, then

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Spin hypersurface of $\mathbb{C}P^n$

$X \xrightarrow{P,D} d \cdot x \in H^2(\mathbb{C}P^n)$, $n \equiv 2 \pmod{4}$, $\alpha(X) \equiv \binom{n+k}{n} \pmod{2}$.

α -invariant: Examples

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- ♣ Example: $\mathbf{I} \neq 0, d_2 = 1, H_{n_1, n_2}^{\mathbf{I}}(d_1, 1)$ spin,

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- ♣ More generally, n_2 even, $d_2 = 1, \alpha(H_{n_1, n_2, \mathbf{I}}(d_1, 1)) = 0$.

Corollary

n_2 even, $d_2 = 1$, then $H_{n_1, n_2}^{\mathbf{I}}(d_1, 1)$ always carries a Riemannian metric of positive scalar curvature. In particular, Milnor hypersurfaces always carries a Riemannian metric of positive scalar curvature.

α -invariant: Positive Scalar Curvature (PSC)

Corollary[The characterisation of PSC ($n_1 = 1$)]

$n_1 + n_2 \equiv 2 \pmod{4}$, $H_{1,n_2}^{\mathbf{I}}(d_1, d_2)$ spin, then there does not exist PSC iff one of the following satisfied

- $k_2 \geq 0, k_2 \equiv 0 \pmod{4}, k_1 \equiv 0 \pmod{2}, \forall i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1$.
- $k_2 \geq 0, k_2 \equiv 1 \pmod{4}, \sigma_1 \equiv 1 \pmod{2}, \forall i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1$.
- $k_2 \geq 0, k_2 \equiv 2 \pmod{4}, k_1 + \sigma_1 \equiv 0 \pmod{2}, \forall i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1$.
- $k_2 \leq -n_2 - 1, k_2 \equiv 0 \pmod{4}, k_1 \equiv 0 \pmod{2}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i([\frac{n_2}{4}]) \leq 1$.
- $k_2 \leq -n_2 - 1, k_2 \equiv 2 \pmod{4}, k_1 + \sigma_1 \equiv 0 \pmod{2}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i([\frac{n_2}{4}]) \leq 1$.
- $k_2 \leq -n_2 - 1, k_2 \equiv 3 \pmod{4}, \sigma_1 \equiv 1 \pmod{2}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i([\frac{n_2}{4}]) \leq 1$.

Corollary[The characterisation of PSC ($n_1 = 2$)]

$n_1 + n_2 \equiv 2 \pmod{4}$, $H_{2,n_2}^I(d_1, d_2)$ spin, not exist PSC iff one of the following satisfied

- $k_2 \geq 0, k_2 \equiv 0 \pmod{4}, k_1 \equiv 0 \text{ or } 1 \pmod{4}, \forall i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 \geq 0, k_2 \equiv 1 \pmod{4}, \binom{k_1+2}{2} + \frac{\sigma_1^2 - 2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2} \equiv 1 \pmod{2}, \forall i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 \geq 0, k_2 \equiv 2 \pmod{4}, \binom{k_1+2}{2} + \sigma_1^2 - \sigma_2 \equiv 1 \pmod{2}, \forall i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 \geq 0, k_2 \equiv 3 \pmod{4}, \binom{k_1+2}{2} + \frac{\sigma_1(2k_1+3-\sigma_1)}{2} \equiv 1 \pmod{2}, \forall i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 = -n_2 - 1, \frac{(k_1+1)(k_1+2)}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \pmod{2}.$
- $k_2 \leq -n_2 - 2, k_2 \equiv 0 \pmod{4}, k_1 \equiv 0 \text{ or } 1 \pmod{4}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 \leq -n_2 - 2, k_2 \equiv 1 \pmod{4}, \binom{k_1+2}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \pmod{2}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 \leq -n_2 - 2, k_2 \equiv 2 \pmod{4}, \binom{k_1+2}{2} + \sigma_1^2 - \sigma_2 \equiv 1 \pmod{2}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i(\frac{n_2}{4}) \leq 1.$
- $k_2 \leq -n_2 - 2, k_2 \equiv 3 \pmod{4}, \binom{k_1+2}{2} + \frac{(2k_1+3)\sigma_1}{2} - \sigma_2 \equiv 1 \pmod{2}, \forall i, a_i([\frac{-k_2-1}{4}]) - a_i(\frac{n_2}{4}) \leq 1.$

\widehat{A} -genus

Theorem

$$\widehat{A}(H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)) = F_{n_1, n_2, \mathbf{I}}(d_1, d_2) - F_{n_1, n_2, \mathbf{I}}(-d_1, -d_2).$$

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Generalized \widehat{A} -genus: the linear sum of $F_{n_1, n_2, \mathbf{I}}(-, -)$.

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For example, let $M = H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$,

$$\begin{aligned} \widehat{A}(M, T_{\mathbb{C}} M) &= \langle \widehat{A}(V) \cdot ch(T_{\mathbb{C}} M), [M] \rangle \\ &= \langle \widehat{A}(V) \cdot \left\{ (n_1 + 1)e^u + e^v + \sum_{j=1}^{n_2} e^{v-i_j u} - e^{d_1 u + d_2 v} \right\}, [M] \rangle \\ &= (n_1 + 1) \langle \widehat{A}(V) e^u, [M] \rangle + \langle \widehat{A}(V) e^v, [M] \rangle - \langle \widehat{A}(V) e^{d_1 u + d_2 v}, [M] \rangle + \sum_{j=1}^{n_2} \langle \widehat{A}(V) \cdot e^{v-i_j u}, [M] \rangle \\ &= (n_1 + 1) F_{n_1, n_2, \mathbf{I}}(2, 0) + F_{n_1, n_2, \mathbf{I}}(0, 2) - F_{n_1, n_2, \mathbf{I}}(2d_1, 2d_2) + \sum_{j=1}^{n_2} F_{n_1, n_2, \mathbf{I}}(-2i_j, 2). \end{aligned}$$

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$$\widehat{A}(H_{n_1, n_2}^{\mathbf{0}}(d_1, d_2)) = (1 - (-1)^{n_1 + n_2}) \binom{n_1 + k_1}{n_1} \binom{n_2 + k_2}{n_2}.$$

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♣ Example: $\mathbf{I} = 0, d_1 = d_2 = 1, H_{n_1, n_2}^{\mathbf{0}}(1, 1)$ Milnor hypersurfaces.

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Corollary [Hirzebruch 1994]

$$\widehat{A}(H_{n_1, n_2}^{\mathbf{0}}(1, 1)) = 0.$$

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- ♣ Example: $\mathbf{I} = 0, d_1 = d_2 = 1, H_{n_1, n_2}^{\mathbf{0}}(1, 1)$ Milnor hypersurfaces.

Corollary [Hirzebruch 1994]

$$\widehat{A}(H_{n_1, n_2}^{\mathbf{0}}(1, 1)) = 0.$$

- ♣ Example: $\mathbf{I} \neq 0, d_2 = 1, n_2$ even,

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\widehat{A} -genus: group action

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If $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$ spin, $F_{n_1, n_2, \mathbf{I}}(d_1, d_2) - F_{n_1, n_2, \mathbf{I}}(-d_1, -d_2)$ non-vanished, then there does not exist non-trivial S^1 action on $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$.

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$\widehat{A}(H_{2, n_2}^{(j, 0, \dots, 0)}(1, n_2 + 1)) \neq 0 \Leftrightarrow j \neq 0, -2$.

Note, $H_{2, n_2}^{(j, 0, \dots, 0)}(1, n_2 + 1)$ spin for j even.

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Problem

Does there exist a non-trivial circle action on $H_{n_1, n_2}^{\mathbf{I}}(d_1, 1)$ for $\mathbf{I} \neq 0$?

Compare H_{n_1, n_2} and $H_{n_1, n_2}^I(d_1, d_2)$

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String manifolds in twisted Milnor hypersurfaces

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Proposition

If $c_1(V) \equiv d_1u + d_2v \pmod{2}$; $p_1(V) = (d_1u + d_2v)^2$, i.e.

$$\begin{cases} n_1 + 1 + \sigma_1 \equiv d_1 \pmod{2}; \\ n_2 + 1 \equiv d_2 \pmod{2}; \\ n_1 + 1 + \sum i_j^2 = d_1^2; \\ n_2 + 1 = d_2^2; \\ \sigma_1 + d_1d_2 = 0. \end{cases}$$

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Theorem

If $n_1 \geq 3, n_2 \geq 3$, there does not exist string manifolds in $H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2)$.

♣ For given n_1, n_2, \mathbf{I} ,

$$H(d_1, d_2; \dots; d_{2k-1}, d_{2k}) := H_{n_1, n_2}^{\mathbf{I}}(d_1, d_2) \bigcap \dots \bigcap H_{n_1, n_2}^{\mathbf{I}}(d_{2k-1}, d_{2k}).$$

call $H(d_1, d_2; \dots; d_{2k-1}, d_{2k})$ **Twisted complete intersections**.

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Convention

$H(d_1, d_2; \dots; d_{2k-1}, d_{2k})$ string means it satisfies condition 1.

Note If \mathbf{I} negative, d_1, d_2 positive, based on Lefschetz hyperplane theorem, as long as $n_1 + n_2 > 3$ the two **sufficient conditions** mentioned before are **sufficient and necessary conditions**.

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Lemma

given $n_1 > 0, n_2 > 0, \mathbf{I}, d_1, d_2, n_2 \equiv d_2 + 1 \pmod{2}$, then

$$F_{n_1, n_2, \mathbf{I}}(d_1, d_2) = \begin{cases} 0, & |d_2| < n_2, \\ \binom{\frac{d_1+n_1-1+\sigma_1}{2}}{n_1}, & d_2 = n_2 + 1, \\ (-1)^{n_2} \binom{\frac{d_1+n_1-1-\sigma_1}{2}}{n_1}, & d_2 = -(n_2 + 1). \end{cases} \quad (2)$$

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Theorem

$H(d_1, d_2; \dots; d_{2k-1}, d_{2k})$ string, then

$$\widehat{A}(H(d_1, d_2; \dots; d_{2k-1}, d_{2k})) = 0.$$

* Binomial numbers

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$$A(n, m), n, m \in \mathbb{Z}_+$$

Recurrence: $A(n, m) = \frac{m}{n}(A(n - 1, m) + A(n - 1, m - 1)).$

Explicit formula: $A(n, m) = \frac{m}{n!} \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} l^n \quad (n \in \mathbb{Z}_+).$

Generating function: $e^{y(e^x - 1)} - 1 = \sum_{m,n \geq 0} A(n, m) x^n \frac{y^m}{m!}.$

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proposition

- ① $S(n, l) = \frac{n!}{l!} A(n, l).$ ($S(n, l)$ the Stirling number of the second kind.)
- ② $B_n = \sum_{l=0}^n \frac{n!}{l!} A(n, l).$ (B_n is the Bell number.)
- ③ $B_n(0) = \sum_{l=0}^n \frac{n!}{l+1} A(n, l).$ ($B_n(t)$ the Bernoulli number)
- ④ $P_n[0, 1, 2, \dots, l] = \frac{n!}{l!} A(n, l)$ (P_n : l^{th} difference for $P_n(x) = x^n.$)

* Generalised Bott manifold

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For any generalised Bott manifold M over $\Delta^{n_0} \times \cdots \times \Delta^{n_r}$ with

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \ddots & & \vdots & \\ 0 & & 1 & -1 & \\ & & a_1^1 & 1 & -1 \\ & & \vdots & \ddots & \vdots \\ & & a_{n_1}^1 & 1 & -1 \\ & & \vdots & & \vdots \\ & & a_{\sum n_i}^1 & a_{\sum n_i}^2 & a_{n_r}^r & 1 & -1 \\ & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & a_{\sum n_i}^1 & a_{\sum n_i}^2 & a_{n_r}^r & 1 & -1 \end{bmatrix}$$

Let H be a spin submanifold of M , $H \xleftarrow{P.D} d_1u + d_2v$. Let $n = (n_0, \dots, n_r)$ and the rest notations are the same.

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Theorem

$$\widehat{A}(H) = F_{n_1, n_2, \mathbf{I}}(d_1, d_2) - F_{n_1, n_2, \mathbf{I}}(-d_1, -d_2) \text{ for any integers } d_1, d_2.$$

$$\alpha(H) \equiv F_{n_1, n_2, \mathbf{I}}(d_1, d_2) \pmod{2} \text{ for } H \text{ spin and } \sum_{i=0}^r n_i \equiv 2 \pmod{4}.$$

Where $F_{n, \mathbf{I}}(d_1, d_2) =$

$$\sum_{\substack{1 \leq p \leq r \\ 1 \leq i_p \leq n_p \\ 0 \leq l_{i_p} \leq n_p \\ 0 \leq m_{i_p} \leq l_{i_p}}} \binom{l}{m} (-1)^{l-m} \binom{n_0 + k_0 + m_1 a^1}{n_0} \binom{n+k+ma}{n+l}_{r-1} \binom{n_r + k_r}{n_r + l_r},$$

$$\text{and } l = \sum_{p=1}^r \sum_{i_p=1}^{n_p} l_{i_p}, m = \sum_{p=1}^r \sum_{i_p=1}^{n_p} m_{i_p}, m_p a^p = \sum_{i_p=1}^{n_p} m_{i_p} a_{i_p}^p,$$

$$\binom{n+k+ma}{n+l}_{r-1} = \binom{n_1+k_1+m_1 a^1}{n_1+l_1} \cdots \binom{n_{r-1}+k_{r-1}+m_{r-1} a^{r-1}}{n_{r-1}+l_{r-1}}.$$

Ongoing works

- * Witten genus
- * elliptic genus(real)
- * elliptic genus(complex)
- * ...

Thank you for your listening

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