Real Lagrangians in symplectic toric del Pezzo surfaces

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Symplectic manifolds

Definition

A **symplectic manifold** is a smooth manifold M equipped with a closed nondegenerate 2-form ω called a symplectic form.

- A symplectic form measures a signed area of surfaces in M.

Example

- $(\mathbb{C}^n, \omega_{\text{std}} = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j)$
- $(S^2,$ any area form$)$, surfaces with area form
- $(S^2 \times S^2, \omega \oplus \omega)$, product of two symplectic manifolds.

- Every symplectic manifold has even dimension.

Lagrangian submanifolds

Let (M^{2n}, ω) be a symplectic manifold.

Definition

A submanifold L of M is called **Lagrangian** if $\dim L = n$ and $\omega|_{TL} \equiv 0$.

- Maximally degenerated submanifold.
- How to find Lagrangian submanifolds?

Definition

- An anti-symplectic involution R of M is a diffeomorphism of M satisfying $R^2 = id_M,\ R^*\omega = -\omega$.
- The fixed point set of an antisymplectic involution is a Lagrangian submanifold. We call such a Lagrangian is **real**.

Examples

Consider $(S^2, \omega = d\theta \wedge dh)$ with $H(\theta, h) = h$.

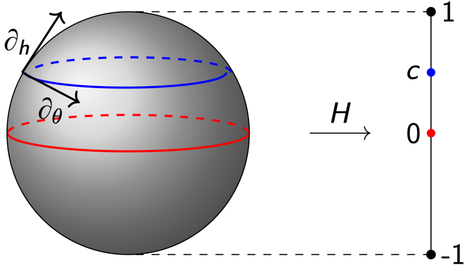


Figure: A height function on a symplectic sphere

- $H^{-1}(c) \cong S^1$: Lagrangian for each $c \in (-1, 1)$
- $H^{-1}(0)$ (equator) : real Lagrangian ($\because R(\theta, h) = (\theta, -h)$)

Symplectic toric manifolds

We denote $\mathfrak{t}$ by the Lie algebra of a torus $\mathbb{T}$.

Definition (Hamiltonian $\mathbb{T}$-action on (M, ω))

A symplectic $\mathbb{T}$-action on (M, ω) is **Hamiltonian** if there is a map $\mu \colon M \to \mathfrak{t}^*$ such that

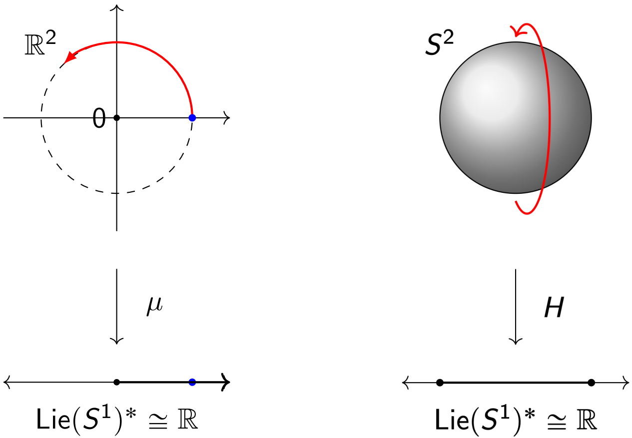
- $\omega(X_\xi, \cdot) = -d\langle \mu, \xi \rangle$, X_ξ : the induced vector field of $\xi \in \mathfrak{t}$.
- μ : invariant under the $\mathbb{T}$-action.

We call μ a **moment map** of the Hamiltonian $\mathbb{T}$-action.

Definition

- A **symplectic toric manifold** is a (compact connected) symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian $\mathbb{T}^n$-action and a moment map $\mu \colon M \to \mathfrak{t}^*$.
- The image $\mu(M)$ of a moment map is called the moment polytope.

Examples of symplectic toric manifolds



(a) $(\mathbb{R}^2, dx \wedge dy)$

(b) $(S^2, d\theta \wedge dh)$

Figure: Examples of symplectic toric manifolds and moment polytopes.

Delzant theorem

Theorem (Delzant,'90)

The map

$$(M^{2n}, \omega, \mu) \mapsto \mu(M)$$

gives a bijection from the set of symplectic toric manifolds onto the set of polytopes satisfies following properties:

1. *There are n edges meeting at each vertex.*
2. *Each edges meeting at vertex has rational slope.*
3. *The set of slopes of edges meeting at vertex is a $\mathbb{Z}$-basis.*

Such a polytope is called a **Dezant polytope.**

Meaning

Combinatorial perspective to the study of symplectic toric manifolds.

Equivariant antisymplectic involutions

(M, ω, μ) : symplectic toric manifold with $\Delta := \mu(M)$.

Definition (Equivariant antisymplectic involution)

An antisymplectic involution R of (M, ω, μ) is **equivariant** if there is a group involution $R_{\mathbb{T}}$ of a torus $\mathbb{T}$ such that

$$R(t \cdot x) = R_{\mathbb{T}}(t) \cdot R(x)$$

for all $x \in M$ and $t \in \mathbb{T}$.

- Symmetry of Δ : lattice preserving automorphism σ of $\mathfrak{t}^*$ satisfying $\sigma(\Delta) = \Delta$.
- An equvariant antisymplectic involution induces a symmetry σ of Δ such that $\sigma^2 = id$ and $\mu \circ R = \sigma \circ \mu$.

Main theorem

Question

For a given involution σ of Δ,

- Find an equivariant antisymplectic involution R^σ and $R^\sigma_{\mathbb{T}}$.
- Reconstruct the real Lagrangian $\mathrm{Fix}(R^\sigma)$ from the data of the $\mathrm{Fix}(\sigma) \subset \Delta$.

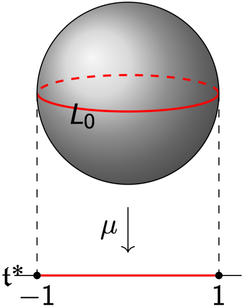
Theorem (Brendel-Kim-M)

Let (M, ω, μ) be a symplectic toric manifold and let σ be an involution on moment polytope $\Delta = \mu(M)$. Then

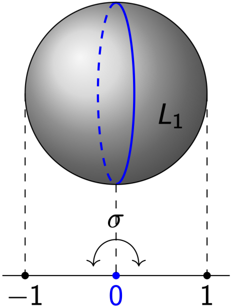
- *The antisymplectic involution R^σ is induced by complex conjugation of $(\mathbb{C}^k, \omega_{std})$ and $\sigma^* = -(R^\sigma_{\mathbb{T}})^{-1}$.*
- *Roughly, $Fix(R^\sigma)/Fix(R^\sigma_{\mathbb{T}}) = Fix(\sigma)$.*

Real Lagrangians in a symplectic toric sphere

Consider $(S^2, \omega, \mu\colon S^2 \to \mathfrak{t}^*)$ with real lagrangians.



(a) $\mu(S^2) = \mu(L_0)$



(b) $\mu(S^2) \neq \mu(L_1)$

Figure: The fixed point set $\mu(L_1) = \{0\}$ of the symmetry σ on $[-1, 1]$

The main application

Theorem (Brendel-Kim-M)

Let M be a toric symplectic del Pezzo surface and let L be a real Lagrangian of M. The diffeomorphism type of L is given as follows.

M	$L = Fix(R)$					
$S^2 \times S^2$	S^2	T^2				
X_0			$\mathbb{R}P^2$			
X_1				$\mathbb{R}P^2 \# \mathbb{R}P^2$		
X_2			$\mathbb{R}P^2$		$\#_3 \mathbb{R}P^2$	
X_3	S^2	T^2		$\mathbb{R}P^2 \# \mathbb{R}P^2$		$\#_4 \mathbb{R}P^2$

Toric symplectic del Pezzo surfaces

Definition

A symplectic manifold (M, ω) is **monotone** if $c_1(M) = \kappa \cdot [\omega]$ for some $\kappa > 0$.

Theorem (Li-Liu, Ohta-Ono, McDuff, Taubes)

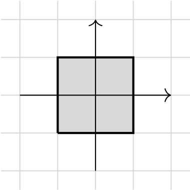
Let (M, ω) be a monotone closed symplectic 4-manifold. Then

- $M \cong S^2 \times S^2$ *or* $X_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $0 \leqslant k \leqslant 8$.
- *The uniqueness of the monotone symplectic structure* ω.

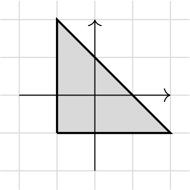
Such a (M, ω) *is called a* **symplectic del Pezzo surface**.

Theorem

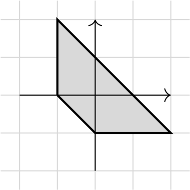
A monotone toric structure exists only on $S^2 \times S^2$ *and* X_k *for* $k = 0, 1, 2, 3$, *which are called* **toric symplectic del Pezzo surfaces**.



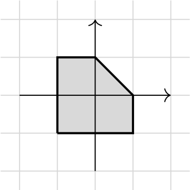
(a) $S^2 \times S^2$



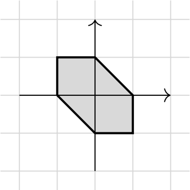
(b) X_0



(c) X_1



(d) X_2



(e) X_3

Figure: Moment polytopes of toric symplectic del Pezzo surfaces

Obstructions

Lemma

If L is an orientable Lagrangian of symplectic del Pezzo surfaces, then $L \cong S^2$ or T^2.

Theorem (Smith inequality, Euler characteristic relation, '40)

Let $I: X \to X$ be an involution on a manifold X. Then we have

$$dim H_*(X;\mathbb{Z}_2) \geqslant dim H_*(Fix(I);\mathbb{Z}_2), \ \chi(X) \equiv \chi(Fix(I)) \ mod\ 2.$$

where χ denotes the Euler characteristic.

Lemma (Arnold, '71)

For an antisymplectic involution R on a symplectic manifold M^4,

$$[Fix(R)] \cdot R_*(\alpha) = \alpha \cdot R_*(\alpha), \ for\ all\ \ \alpha \in H_2(M;\mathbb{Z}_2).$$

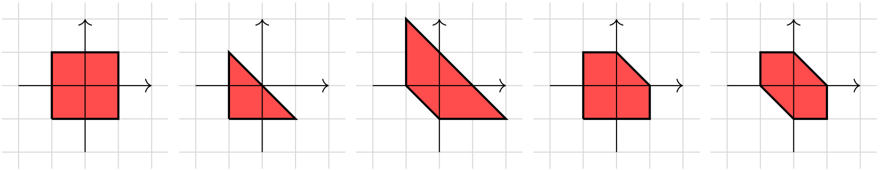
Candidates of real Lagrangians

We have a table of the candidates of the diffeomorphism types of real Lagrangians L of a toric symplectic del Pezzo surface M.

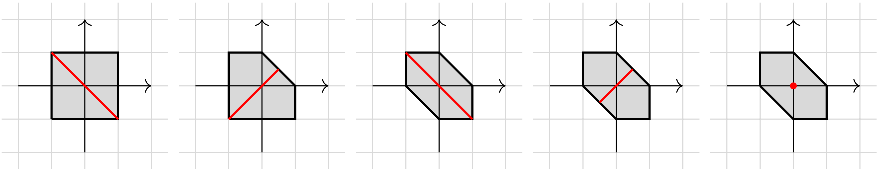
M	$L = \mathrm{Fix}(R)$					
$S^2 \times S^2$	S^2	T^2				
X_0			$\mathbb{R}P^2$			
X_1				$\mathbb{R}P^2 \# \mathbb{R}P^2$		
X_2			$\mathbb{R}P^2$		$\#_3 \mathbb{R}P^2$	
X_3	S^2	T^2		$\mathbb{R}P^2 \# \mathbb{R}P^2$		$\#_4 \mathbb{R}P^2$

We do not know which one can be realized. Hence, we shall construct antisymplectic involutions of M whose fixed set is what we want.

Existence of diffeomorphism types of real Lagrangians



(a) T^2 in Q (b) $\mathbb{R}P^2$ in X_0 (c) $\#_2\mathbb{R}P^2$ in X_1 (d) $\#_3\mathbb{R}P^2$ in X_2 (e) $\#_4\mathbb{R}P^2$ in X_3



(f) S^2 in Q (g) K in X_1 (h) S^2 in X_2 (i) K in X_3 (j) T^2 in X_3

Thank you for attention