Regular semisimple Hessenberg varieties whose cohomology rings are generated by elements of degree 2

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Hessenberg varieties

$[n] = \{1, 2, 3, \ldots, n\}$

$h\colon [n] \to [n]$ is called a Hessenberg function if it satisfies

$$h(1) \le h(2) \le \cdots \le h(n) \quad \text{and} \quad h(i) \ge i \text{ for } \forall i.$$

A Hessenberg variety $\mathrm{Hess}(X, h)$ is a subvariety of a flag variety

$$\mathrm{Flag}(\mathbb{C}^n) = \{V_\bullet = (V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim V_i = i \text{ for } \forall i\},$$

where X is an $n \times n$-matrix and h is a Hessenberg function.

$$\mathrm{Hess}(X, h) = \{(V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \in \mathrm{Flag}(\mathbb{C}^n) \mid XV_i \subset V_{h(i)} \text{ for } \forall i\}$$

We regard an $n \times n$-matrix X as a linear hom. $\mathbb{C}^n \to \mathbb{C}^n$.

Examples of Hessenberg varieties

$$\mathrm{Hess}(X,h) = \{(V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \in \mathrm{Flag}(\mathbb{C}^n) \mid XV_i \subset V_{h(i)} \text{ for } \forall i\}$$

We express h as a vector $(h(1), h(2), \ldots, h(n))$.

When $h = (n, n, \ldots, n)$, $\mathrm{Hess}(X,h) = \mathrm{Flag}(\mathbb{C}^n)$ for any X.

When X has distinct eigenvalues, X is called regular semisimple.
In this case, we write S instead of X.

$\mathrm{Hess}(S,h)$ is called a regular semisimple Hessenberg variety.

$$\mathrm{Hess}(S,h) \cong \mathrm{Hess}(S',h)$$

When $h = (2, 3, 4, \ldots, n, n)$, $\mathrm{Hess}(S,h)$ is the permutohedral variety.

When $h = (2, 3, 4, \ldots, n, n)$ and N is regular nilpotent, $\mathrm{Hess}(N,h)$ is the Peterson variety.

Previous results and today's question

Problem.

- Describe the cohomology ring $H^*(\mathrm{Hess}(S,h))$ explicitly.

Previous results on $H^*(\mathrm{Hess}(S,h))$.

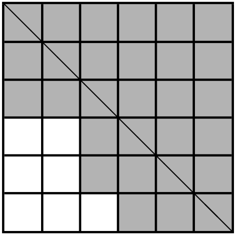
- $\mathfrak{S}_n$-representation structure by Brosnan and Chow (2018)
- a recurrence formula to construct a basis as a module by Chow, Hong, and Lee (arXiv:2107.00863)
- explicit description as a ring for $h = (m, n, \cdots, n)$ by Abe, Horiguchi, and Masuda (2019)

Questions.

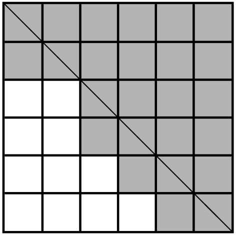
1. When $H^*(\mathrm{Hess}(S,h))$ is generated by H^2?
2. What are generators and relations in that case?

Visualization of Hessenberg functions

It is useful to visualize a Hessenberg function h as follows.



$h = (3, 3, 5, 6, 6, 6)$



$h = (2, 2, 4, 5, 6, 6)$

Inclusion of these pictures means inclusion of Hessenberg var.'s.

When $h(i) = i$ for some $i < n$,
$\mathrm{Hess}(S, h) \cong \bigsqcup \binom{n}{i}$ copies of $\mathrm{Hess}(S_1, h_1) \times \mathrm{Hess}(S_2, h_2)$.

We assume that $\mathrm{Hess}(S, h)$ is connected $\Leftrightarrow$ $h(i) > i$ for $i < n$.

GKM theory

Equivariant cohomology is useful to describe elements of $H^*(\mathrm{Hess}(S,h))$.

$T = \{$diagonal regular $n \times n$-matrices$\} \curvearrowright \mathrm{Flag}(\mathbb{C}^n)$

A regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ has a T-action as the restriction of this action on $\mathrm{Flag}(\mathbb{C}^n)$.

The fixed point set is $\{(\langle \boldsymbol{e}_{w(1)}\rangle \subset \langle \boldsymbol{e}_{w(1)}, \boldsymbol{e}_{w(2)}\rangle \subset \cdots \subset \mathbb{C}^n) \mid w \in \mathfrak{S}_n\}$.

When $T \curvearrowright Y$, $H_T^*(Y) = H^*(ET \times_T Y)$ the equivariant cohomology.

A regular semisimple Hessenberg variety has paving and is equivariantly formal.

$$H_T^*(\mathrm{Hess}(S,h)) \to H_T^*(\mathrm{Hess}(S,h)^T) \cong \bigoplus_{w\in\mathfrak{S}_n} H^*(BT) = \mathrm{Map}(\mathfrak{S}_n, H^*(BT))$$

This restriction is injective.

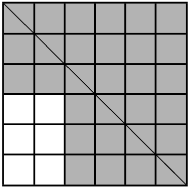
$H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$

$$H^*(\mathrm{Hess}(S,h)) \cong H_T^*(\mathrm{Hess}(S,h))/(H^2(BT))$$

Today's Hessenberg functions

Theorem 1 (Ayzenberg-Masuda-S.)

If $H^*(\mathrm{Hess}(S, h))$ *is generated by the second cohomology, then* $h = (\underbrace{a+1, \ldots, a+1}_{a}, \underbrace{a+2, a+3, \ldots, a+m+1}_{m}, n, \ldots, n)$.

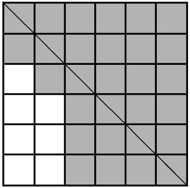


$n = 6$, $a = 2$, $\underline{m = 0}$

$h = (3, 3, 6, 6, 6, 6)$

rectangle case

(removed maximal one)

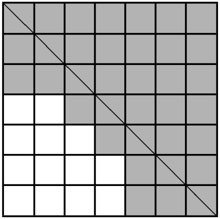


$(n = 6,\ a = 1,\ m = 1)$

$n = 6$, $\underline{a = 0}$, $m = 2$

$h = (2, 3, 6, 6, 6, 6)$

lollipop case



$n = 7$, $a = 2$, $m = 2$

$h = (3, 3, 4, 5, 7, 7, 7)$

double lollipop case

Generators

Lemma 2 (cf. our paper arXiv:2203.11580)

For $h = (\underbrace{a+1,\dots,a+1}_{a},\underbrace{a+2,a+3,\dots,a+m+1}_{m},n,\dots,n)$,

$H_T^*(\mathrm{Hess}(S,h))$ *is generated by* $H^2(BT)$ *and the following elements:*
$x_i - x_{i+1}$, y_i, *and* τ_A *for* $i \in [n]$ *and* $A \subset [n]$ *with* $a < |A| \le a+m$.

They are defined by

$$x_i(w) = t_{w(i)} \quad (w \in \mathfrak{S}_n)$$

$$y_i(w) := y_{a,i}(w) = \begin{cases} t_i - t_{w(a+1)} & i \in \{w(1),\dots,w(a)\} \\ 0 & \textit{otherwise} \end{cases}$$

$$\tau_A(w) = \begin{cases} t_{w(|A|)} - t_{w(|A|+1)} & A = \{w(1),\dots,w(|A|)\} \\ 0 & \textit{otherwise} \end{cases}$$

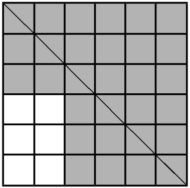
$$\left(y_i^*(w) := y^*_{a+2,i}(w) = \begin{cases} t_{w(a+1)} - t_i & i \in \{w(a+2),\dots,w(n)\} \\ 0 & \textit{otherwise} \end{cases} \right)$$

We have $y_i^* = x_{a+1} - t_i + y_i$.

Relations

Let x_i, y_i, and τ_A denote the corresponding ordinary cohomology class also. $H_T^*(\mathrm{Hess}(S,h)) \to H^*(\mathrm{Hess}(S,h))$

Assume that $a < \frac{n}{2}$.



Lemma 3

In the rectangle case $h = (a+1, \ldots, a+1, n, \ldots, n)$, *we have the following relations in* $H^*(\mathrm{Hess}(S,h))$.

Let $I \subset [n]$, *and define* $y_I = \prod_{i \in I} y_i$ *and* $y_I^* = \prod_{i \in I} y_i^*$.

(0) $e_i(x_1, \ldots, x_n) = 0 \quad (1 \leq i \leq n)$ *come from* $\mathrm{Flag}(\mathbb{C}^n)$

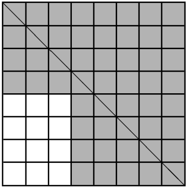
(1) $y_I = 0 \quad (|I| = a+1)$

(1*) $y_I^* = 0 \quad (|I| = n-a)$

(2) $\displaystyle\sum_{|I|=r} y_I - \sum_{j=0}^{r} \binom{a-j}{r-j} e_j(x_1, \ldots, x_a)(-x_{a+1})^{r-j} = 0 \quad (1 \leq r \leq a)$

(3) $y_i^2 + x_{a+1} y_i = 0 \quad (1 \leq i \leq n)$

Poincaré series



(2) are deformations of the relations of $\mathrm{Flag}(\mathbb{C}^a)$.

$$e_1(x_1,\dots,x_a) = \sum_i y_i + a x_{a+1}$$

$$e_2(x_1,\dots,x_a) = \sum_{|I|=2} y_I + (a-1)e_1(x_1,\dots,x_a)x_{a+1} - \binom{a-2}{2}x_{a+1}^2$$

= a polynomial in y_i's and x_{a+1}

$$e_3(x_1,\dots,x_a) = \cdots$$

Lemma 4

For $h = (a+1,\dots,a+1,n,\dots,n)$,

$$\mathrm{Poin}(\mathrm{Hess}(S,h),\sqrt{q}) = [a]_q![n-a-1]_q! \sum_{k=0}^{a} \binom{n}{k} q^k [n-2k]_q,$$

where $[m]_q = \dfrac{1-q^m}{1-q} = 1 + q + q^2 + \cdots + q^{m-1}$ *and* $[m]_q! = \prod_{k=1}^{m} [k]_q$.

Remark that $[m]_q! = \mathrm{Poin}(\mathrm{Flag}(\mathbb{C}^m),\sqrt{q})$.

The structure of $H^*(\mathrm{Hess}(S,h))$ in the rectangle case

Since x_i's and y_i's are generators, we have a surjection.

$$\mathbb{Q}[X_i, Y_i \mid i \in [n]] \to H^*(\mathrm{Hess}(S,h)), \quad X_i \mapsto x_i,\ Y_i \to y_i$$

Let $\mathfrak{I}$ be the ideal of $\mathbb{Q}[X_i, Y_i \mid i \in [n]]$ generated by the followings:

(0) $e_i(X_1, \ldots, X_n) \quad (1 \le i \le n)$

(1) $Y_I \quad (|I| = a+1)$

(1*) $Y_I^* \quad (|I| = n-a)$

(2) $\displaystyle\sum_{|I|=r} Y_I - \sum_{j=0}^{r} \binom{a-j}{r-j} e_j(X_1, \ldots, X_a)(-X_{a+1})^{r-j} \quad (1 \le r \le a)$

(3) $Y_i^2 + X_{a+1} Y_i \quad (1 \le i \le n)$

where $Y_I = \prod_{i \in I} Y_i$ and $Y_I^* = \prod_{i \in I} (Y_i + X_{a+1})$.

Theorem 5 (Ayzenberg-Masuda-S.)

$\mathbb{Q}[X_i, Y_i \mid i \in [n]]/\mathfrak{I} \to H^*(\mathrm{Hess}(S,h))$ *is an isomorphism.*

Outline of the proof

Compare $\mathrm{Hilb}(\mathbb{Q}[X_i, Y_i \,|\, i \in [n]]/\mathfrak{I}, \sqrt{q})$ and $\mathrm{Poin}(\mathrm{Hess}(S, h), \sqrt{q})$.

For any element of $\mathbb{Q}[X_i, Y_i \,|\, i \in [n]]/\mathfrak{I}$, find a good representative.

We can find it as a linear combination of

$$\left\{ X_{a+1}^{i_{a+1}} X_1^{i_1} \cdots X_a^{i_a} X_{a+2}^{i_{a+2}} \cdots X_n^{i_n} Y_I \;\middle|\; \begin{array}{c} |I| \le a,\ 0 \le i_{a+1} \le n, \\ 1 \le k \le a \ \Rightarrow\ 0 \le i_k \le a - k, \\ a+2 \le k \le n \ \Rightarrow\ 0 \le i_k \le n - k \end{array} \right\}$$

by (0), (1), (2), and (3). When $I = \emptyset$, set $Y_\emptyset = 1$.

Lemma 6

Let $r = |I|$. By (1^*), we have

$$X_{a+1}^{n-2r} Y_I = -\sum_{k=0}^{r-1} \frac{\binom{a-k}{r-k}}{\binom{n-r-k}{r-k}} X_{a+1}^{n-r-k} e_k(Y_i \,|\, i \in I)$$

Furthermore

In the lollipop case $h = (2, 3, \ldots, m+1, n, \ldots, n)$,

$$\mathrm{Poin}(\mathrm{Hess}(S,h); \sqrt{q}) = [n-m-1]_q! \left([n]_q + q \sum_{k=1}^{m} \binom{n}{k} A_k(q) [n-k-1]_q \right),$$

where $A_k(q)$ is the Eulerian polynomial of degree k.

$A_k(q) = \mathrm{Poin}(\mathrm{Perm}_k,\ \sqrt{q})$

We have the following relations in $H^*(\mathrm{Hess}(S,h))$:

(0) $e_i(x_1, \ldots, x_n) = 0 \quad (1 \le i \le n)$

(L1) $e_i(x_1, \ldots, x_{|A|})\tau_A = 0 \quad (1 \le i \le |A|)$

(L2) $\tau_A \tau_B = 0 \quad (A \not\subset B$ or $A \not\supset B)$

(L3) $\displaystyle\sum_{|A|=k} \tau_A = x_k - x_{k+1} \qquad (a < k \le m)$

(L4) $\displaystyle\left(\sum_{p \in A \subsetneq B} \tau_A - \sum_{q \in A \subsetneq B} \tau_A \right) \tau_B = 0 \quad (p, q \in B)$

(L5) $\displaystyle\tau_B h_{n-|B|-1}(x_{|B|}, x_{|B|+1}) - \prod_{i \in [n] \setminus B} \left(x_{|B|} + \sum_{i \in A,\, |A| < |B|} \tau_A \right) = 0$

Furthermore...

In the double lollipop case $h = (a+1, \ldots, a+1, a+2, \ldots, a+m+1, n, \ldots, n)$,

$$\mathrm{Poin}(\mathrm{Hess}(S,h); \sqrt{q}) = [a]_q![n-a-m-1]_q! \left(\sum_{k=0}^{a} \binom{n}{k} q^k [n-2k]_q \right.$$

$$\left. + q \sum_{l=1}^{m} \binom{n}{a+l} [n-a-l-1]_q \left([a+l]_q + q \sum_{k=1}^{l-1} \binom{a+l}{k} A_k(q) [a+l-k-1]_q \right) \right)$$

We have the following relations in $H^*(\mathrm{Hess}(S,h))$:
(0), (1), (1*), (2), (3), (L1)–(L5) with $y_{a,i} = \sum_{i \in A, |A| \le a} \tau_A$,

$$\text{(L5}^*\text{)} \quad \tau_C h_{k-1}(x_{|C|}, x_{|C|+1}) - \prod_{i \in C} \left(-y_i - \sum_{i \in A,\, a < |A| \le |C|} \tau_A \right) = 0$$

$$\text{(L6)} \quad \tau_B \left(x_{|B|} + y_i + \sum_{i \in A \subsetneq B,\, a < |A|} \tau_A \right) = 0 \quad (i \in B)$$