On cohomology of (quasi)toric manifolds over a vertex cut of a finite product of simplicies

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Motivation

- Suyong Choi, Mikiya Masuda, and Dong Youp Suh, Quasitoric manifolds over a product of simplices, Osaka J. Math. 47 (2010), no. 1, 109–129. MR 2666127
- Sho Hasui, Hideya Kuwata, Mikiya Masuda, and Seonjeong Park, Classification of toric manifolds over an n-cube with one vertex cut, Int. Math. Res. Not. IMRN (2020), no. 16, 4890–4941. MR 4139029

Quasitoric manifold and its cohomology

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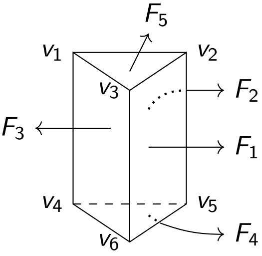
- Let Q be an n-dimensional simple polytope and F a d-dimensional face in Q. Then $F = \bigcap_{j=1}^{n-d} F_{i_j}$ for some unique facets $F_{i_1}, \ldots, F_{i_{n-d}}$ of Q.
- We call $\bigcap_{j=1}^{s} F_{i_j}$ a minimal non-face of Q if

$$\bigcap_{j=1}^{s} F_{i_j} = \varnothing \text{ and } \bigcap_{\substack{j=1\\ j\neq t}}^{s} F_{i_j} \neq \varnothing$$

for some $1 \leqslant t \leqslant s$.



- Let $\lambda\colon \mathcal{F}(Q) \to \mathbb{Z}^n$ be a function such that

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- Each λ_i determines a line in $\mathbb{R}^n$, whose image under $exp\colon \mathbb{R}^n \to T^n = (\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R})/\mathbb{Z}^n$ is a circle subgroup, denoted by T_i.

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- Consider $X(Q, \lambda) := (T^n \times Q)/\sim$,

$$(t, x) \sim (s, y) \text{ if and only if } x = y \in F \text{ and } t^{-1}s \in T_F. \quad (2.2)$$

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The ideal J is generated by the n coordinates of λ_J.

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Example

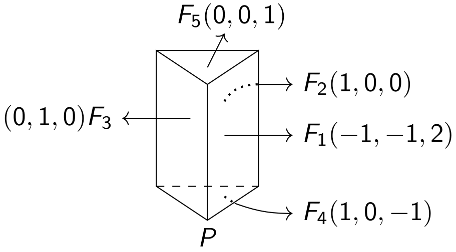
$H^*(X(P,\lambda);\mathbb{Z}) = \mathbb{Z}[x_1,\dots,x_5]/(I+J)$
where $I = \langle x_1x_2x_3, x_4x_5 \rangle$
and

$$\lambda_J = \sum_{j=1}^{5} \lambda_j x_j$$
$$= (-x_1+x_2+x_4, -x_1+x_3+x_5, 2x_1-x_4+x_5).$$

Thus $J =$
$\langle -x_1+x_2+x_4, -x_1+x_3+x_5, 2x_1-x_4+x_5 \rangle$.
This implies

$$H^*(X(P,\lambda);\mathbb{Z}) = \mathbb{Z}[x_1,x_4]/\overline{I}$$

where
$\overline{I} = \langle x_1(x_1-x_4)(3x_1-x_4), x_4(-2x_1+x_4) \rangle$.



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- Let $V(\Delta^{n_j}) := \{v_0^j, \ldots, v_{n_j}^j\}$ and $\mathcal{F}(\Delta^{n_j}) := \{f_0^j, \ldots f_{n_j}^j\}$ where the unique facet $f_{k_j}^j$ does not contain the vertex $v_{k_j}^j$ in Δ^{n_j} for $0 \leqslant k_j \leqslant n_j$ for $j = 1, \ldots, m$.

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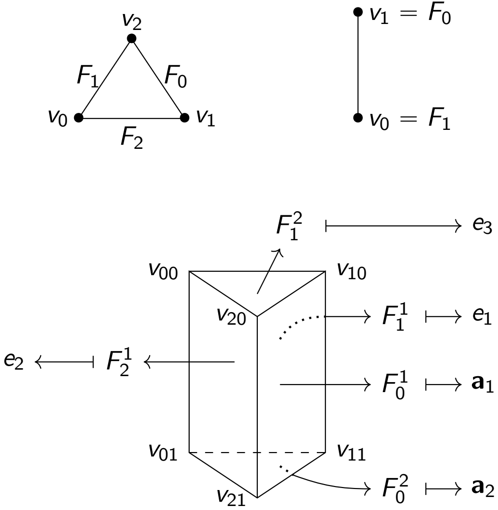
- Let $V(\Delta^{n_j}) := \{v_0^j, \ldots, v_{n_j}^j\}$ and $\mathcal{F}(\Delta^{n_j}) := \{f_0^j, \ldots f_{n_j}^j\}$ where the unique facet $f_{k_j}^j$ does not contain the vertex $v_{k_j}^j$ in Δ^{n_j} for $0 \leqslant k_j \leqslant n_j$ for $j = 1, \ldots, m$.Then
$$V(P) = \{v_{\ell_1\ell_2\ldots\ell_m} := (v_{\ell_1}^1, v_{\ell_2}^2, \ldots, v_{\ell_m}^m) \mid 0 \leqslant \ell_j \leqslant n_j, j = 1, \ldots, m\} \tag{3.2}$$
$$\mathcal{F}(P) = \{F_{k_j}^j \mid 0 \leqslant k_j \leqslant n_j, 1 \leqslant j \leqslant m\}$$
$$\text{where } F_{k_j}^j := \Delta^{n_1} \times \cdots \times \Delta^{n_{j-1}} \times f_{k_j}^j \times \Delta^{n_{j+1}} \times \cdots \times \Delta^{n_m}.$$

- Notice that
  $\mathbf{v}_0 := v_{0\dots 0} = F_1^1 \cap \cdots \cap F_{n_1}^1 \cap \cdots \cap F_1^j \cap \dots F_{n_j}^j \cap \cdots \cap F_1^m \cap \cdots \cap F_{n_m}^m$.
- Let
  $$\lambda \colon \mathcal{F}(P) \to \mathbb{Z}^n \tag{3.3}$$
  be a characteristic function on P where
  $$\begin{aligned} \lambda(F_1^1) = e_1, \dots, \lambda(F_{n_1}^1) = e_{n_1}, \\ \vdots \\ \lambda(F_1^j) = e_{N_{j-1}+1}, \dots, \lambda(F_{n_j}^j) = e_{N_j}, \\ \vdots \\ \lambda(F_1^m) = e_{N_{m-1}+1}, \dots, \lambda(F_{n_m}^m) = e_n. \end{aligned} \tag{3.4}$$
  For the remaining m facets $F_0^1, \dots, F_0^m$, we denote
  $$\mathbf{a}_j := \lambda(F_0^j) \in \mathbb{Z}^n \quad \text{for } j = 1, \dots, m. \tag{3.5}$$

Example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$F_0^1 \longmapsto \mathbf{a}_1$
v_{01}
v_{11}
v_{21}
$F_0^2 \longmapsto \mathbf{a}_2$

Theorem 3.1 ([2])

Let $X(P, \lambda)$ be a quasitoric manifold where $P = \prod_{j=1}^{m} \Delta^{n_j}$ is a product of simplices as in (3.1) and λ is defined following (3.4) and (3.5). Then

$$H^*(X(P,\lambda);\mathbb{Z}) \cong \mathbb{Z}[y_1, \dots, y_m]/\mathcal{L}, \tag{3.6}$$

where the indeterminate y_j is assigned to the facet F_0^j for $j = 1, \dots, m$ and $\mathcal{L}$ is the ideal generated by

$$y_j \prod_{\ell=1}^{n_j} (a_{1\ell}^j y_1 + a_{2\ell}^j y_2 + \cdots + a_{m\ell}^j y_m) \quad \text{for } j = 1, \dots, m.$$

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Toric manifold and Quasitoric manifold

A simplicial complex $\mathcal{K}$ is a set of simplices that satisfies the following

- Every face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$.
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$$\lambda \colon \mathcal{V}(\mathcal{K}) \to \mathbb{Z}^n.$$

We denote $\lambda(v_i) = \lambda_i$ for $i = 1, \ldots, m$.

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We denote $\lambda(v_i) = \lambda_i$ for $i = 1, \ldots, m$. For each $I \in \mathcal{K}$, one can define the following cone

$$C(I) := \{\sum_{v_i \in I} t_i \lambda_i \in \mathbb{R}^n \;\mid\; t_i \in \mathbb{R}_{\geqslant 0} \text{ for all } v_i \in I\}.$$

Definition 4.1

The pair $(\mathcal{K}, \lambda)$ is called a (simplicial) fan of dimension n over $\mathcal{K}$ if it satisfies:

1. For $I = \{v_{i_1}, \ldots, v_{i_k}\} \in \mathcal{K}$, the vectors $\lambda_{i_1}, \ldots, \lambda_{i_k}$ are linearly independent over $\mathbb{R}$.

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If $\bigcup_{I \in \mathcal{K}} C(I) = \mathbb{R}^n$, then the fan $(\mathcal{K}, \lambda)$ is called *complete*.If $\{\lambda_i : v_i \in I\}$ forms a part of a basis of $\mathbb{Z}^n$ for any $I \in \mathcal{K}$, then the fan is called *non-singular*.

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1. For $I = \{v_{i_1}, \ldots, v_{i_k}\} \in \mathcal{K}$, the vectors $\lambda_{i_1}, \ldots, \lambda_{i_k}$ are linearly independent over $\mathbb{R}$.
2. $C(I) \cap C(J) = C(I \cap J)$ for $I, J \in \mathcal{K}$.

If $\bigcup_{I \in \mathcal{K}} C(I) = \mathbb{R}^n$, then the fan $(\mathcal{K}, \lambda)$ is called *complete*.If $\{\lambda_i : v_i \in I\}$ forms a part of a basis of $\mathbb{Z}^n$ for any $I \in \mathcal{K}$, then the fan is called *non-singular*.

Definition 4.2

A complete, non-singular toric variety is called a toric manifold.

[3]William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry

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Definition 4.2

A complete, non-singular toric variety is called a toric manifold.

Proposition 4.3 ([3])

There exists a one to one correspondence between the toric varities of complex dimension n and rational fans of dimension n.

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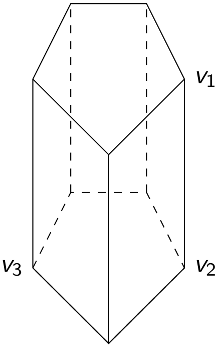
Definition 4.4 (Distance function)

Let P be a polytope and v_1, v_2 two different vertices in P. A *path* between v_1 and v_2 is a sequence of edges $\xi_1, \xi_2, \ldots, \xi_d$ such that $v_1 \in \xi_1$, $v_2 \in \xi_d$ and $\xi_i \cap \xi_{i+1}$ is a vertex of both for $i = 1, \ldots, (d-1)$. The *distance* between two vertices v_1 and v_2 is the minimum d and it is denoted by $D(v_1, v_2)$.



$$D(v_1, v_2) = 1$$
$$D(v_1, v_3) = 3$$
$$D(v_3, v_2) = 2$$
$$D(v_i, v_i) = 0 \ \forall \ i$$

Figure: A distance function on the vertices of a polytope.

- Let (P, λ) be a characteristic pair where P is a product of m simplices and λ a characteristic function on P as defined following (3.4) and (3.5).

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- We fix the order of colums at $\mathbf{v}_0 := v_{0\dots0}$

$$\begin{aligned} A_{\mathbf{v}_0} :&= \begin{pmatrix} \lambda(F_1^1) & \dots & \lambda(F_{n_1}^1) & \dots & \lambda(F_1^m) & \dots & \dots & \lambda(F_{n_m}^m) \end{pmatrix} \\ &= \begin{pmatrix} e_1 & \dots & e_{n_1} & \dots & \dots & e_{N_{m-1}+1} & \dots & e_n \end{pmatrix}. \end{aligned} \tag{4.1}$$

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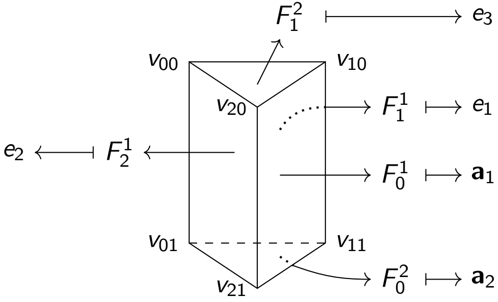

- Let $D(\mathbf{v}, \mathbf{v}_0) = d > 0$. Then we may consider a path of length d from $\mathbf{v}_0$ to $\mathbf{v}$.

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- Let $\mathbf{v} \in V(P)$. Then $\mathbf{v} = v_{\ell_1\ell_2\ldots\ell_m}$ for some $0 \leqslant \ell_j \leqslant n_j$, $j = 1, \ldots, m$ and

$$\mathbf{v} = \bigcap_{\substack{j=1 \\ k_j \neq \ell_j}}^{m} F^j_{k_j}.$$

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- Let $\mathbf{v} \in V(P)$. Then $\mathbf{v} = v_{\ell_1 \ell_2 \ldots \ell_m}$ for some $0 \leqslant \ell_j \leqslant n_j$, $j = 1, \ldots, m$ and

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- If $\ell_j \neq 0$ for $j \in \{1, \ldots, m\}$, then $e_{N_{j-1}+\ell_j}$ is replaced by $\mathbf{a}_j$ by keeping the order of other columns of $A_{\mathbf{v}_0}$ intact.
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- Note that the matrix $A_{\mathbf{v}}$ does not alter by the choice of the path if we choose any other shortest path of length d.
- If $\mathbf{v}$ is a vertex such that $D(\mathbf{v}, \mathbf{v}_0) = m$, i.e., $\ell_j \neq 0$ for all $j = 1, \ldots, m$. Then the matrix $A_{\mathbf{v}}$ is given by

$$A_{\mathbf{v}} = \big(e_1 \ \ldots \ e_{\ell_1 - 1} \ \mathbf{a}_1 \ e_{\ell_1+1} \ \ldots \ e_{N_1} \ e_{N_1+1} \ \ldots \ e_{N_1+\ell_2-1} \ \mathbf{a}_2 \ e_{N_1+\ell_2+1} \ \ldots \\ e_{N_2} \ \ldots \ e_{N_{m-1}+1} \ \ldots \ e_{N_{m-1}+\ell_m-1} \ \mathbf{a}_m \ e_{N_{m-1}+\ell_m+1} \ \ldots \ e_{N_m} \big).$$

Let σ be an n-dimensional nonsingular cone in $\mathbb{R}^n$. Then σ is generated by n linearly independent vectors $\{\xi_1, \ldots, \xi_n\}$ in $\mathbb{R}^n$. Let $M := (\xi_1, \ldots, \xi_n)$ be the nonsingular $n \times n$ matrix. By $\det(\sigma)$ we denote the determinant of the matrix M.

Lemma 4.5

Let σ_1 and σ_2 be two nonsingular cones in $\mathbb{R}^n$ of dimension n. If $\sigma_1 \cap \sigma_2$ is a face of dimension $n-1$ then $\det(\sigma_1)$ and $\det(\sigma_2)$ have differnt signs.

Theorem 4.6

Let P be a finite product of simplices as (3.1) and λ a characteristic function on P as in (3.3). If $X(P, \lambda)$ is a toric manifold then

$$\det A_{\mathbf{v}} = \begin{cases} -1 & \text{if } D(\mathbf{v}, \mathbf{v}_0) = odd \\ +1 & \text{if } D(\mathbf{v}, \mathbf{v}_0) = even \end{cases} \tag{4.2}$$

where $\mathbf{v}_0$ denotes the vertex $v_{0\ldots0}$.

Theorem 4.7

Let P be a product of two simplices and λ a characteristic function defined on P as in (3.3) such that for any vertex $\mathbf{v} \in V(P)$ the following holds:

$$\det A_{\mathbf{v}} = \begin{cases} -1 & \textit{if } D(\mathbf{v}, \mathbf{v}_0) = \textit{odd} \\ +1 & \textit{if } D(\mathbf{v}, \mathbf{v}_0) = \textit{even} \end{cases} \tag{4.3}$$

Then $X(P, \lambda)$ is a toric manifold.

Quasitoric manifold over vertex of product of simplices

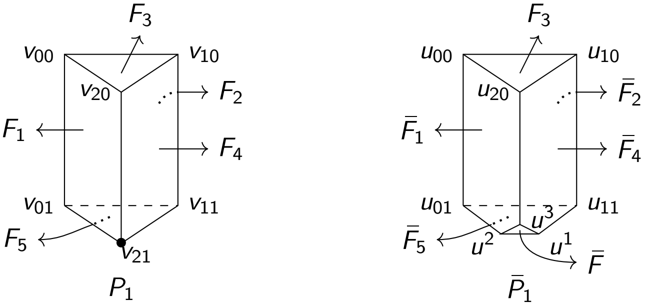

Figure: A vertex cut of a prism where the facets and vertices are induced from Δ^2 and I.

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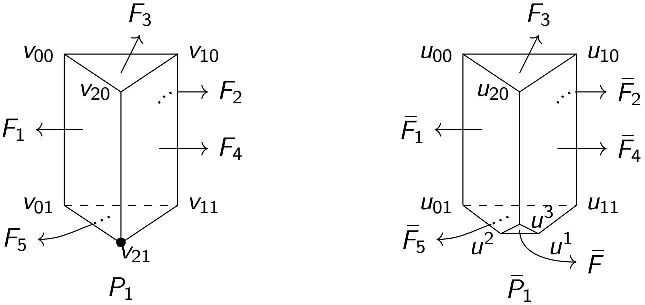

Figure: A vertex cut of a prism where the facets and vertices are induced from Δ^2 and I.

Let $\bar{P}$ be a vertex cut of P at the vertex $\tilde{\mathbf{v}} := v_{n_1 n_2 \dots n_m}$. Then the vertex set and the facet set of $\bar{P}$ are respectively

$$\begin{aligned} V(\bar{P}) &:= (V(P)\setminus\{\tilde{\mathbf{v}}\}) \cup V(\bar{F}), \\ \mathcal{F}(\bar{P}) &:= \{\bar{F}^j_{k_j} := F^j_{k_j} \cap \bar{P} \mid F^j_{k_j} \in \mathcal{F}(P)\} \cup \{\bar{F}\}. \end{aligned} \tag{5.1}$$

Let

$$\overline{\lambda}\colon \mathcal{F}(\bar{P}) \to \mathbb{Z}^n \tag{5.2}$$

be a characteristic function defined as follows

$$\begin{aligned}
\overline{\lambda}(\bar{F}_j^1) &:= e_j \qquad \text{for } j = 1, \ldots, n_1, \\
&\vdots \\
\overline{\lambda}(\bar{F}_j^m) &:= e_{N_{m-1}+j} \quad \text{for } j = 1, \ldots, n_m, \\
\overline{\lambda}(\bar{F}_0^j) &:= \mathbf{a}_j \in \mathbb{Z}^n \quad \text{for } j = 1, \ldots, m, \\
\overline{\lambda}(\bar{F}) &:= \mathbf{b} \in \mathbb{Z}^n.
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where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{Z}^n$.
The characteristic pair $(\bar{P}, \overline{\lambda})$ induces a map $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$ defined by

$$\lambda(F_{k_j}^j) := \overline{\lambda}(\bar{F}_{k_j}^j) \tag{5.4}$$

for $j = 1, \ldots, m$ and $1 \leqslant k_j \leqslant n_j$.

Note that, $\tilde{\mathbf{v}} = F_1^1 \cap \cdots \cap F_{n_1-1}^1 \cap F_0^1 \cap \ldots F_1^m \cap \cdots \cap F_{n_m-1}^m \cap F_0^m$.

Note that, $\tilde{\mathbf{v}} = F^1_1 \cap \cdots \cap F^1_{n_1-1} \cap F^1_0 \cap \dots F^m_1 \cap \cdots \cap F^m_{n_m-1} \cap F^m_0$.
The following matrix

$$A_{\tilde{\mathbf{v}}} := A_{v_{n_1 \dots n_m}} = \begin{pmatrix} e_1 & \dots & e_{N_1-1} & \mathbf{a}_1 & e_{N_1+1} & \dots & \mathbf{a}_{m-1} & e_{N_{(m-1)}+1} & \dots & e_{N_m-1} & \mathbf{a}_m \end{pmatrix} \tag{5.5}$$

is associated to the vertex $\tilde{\mathbf{v}} \in V(P)$.

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Lemma 5.1 (Sarkar, Sau)

Let $X(\bar{P}, \bar{\lambda})$ be a (quasi)toric manifold where $\bar{P}$ is vertex cut at $\tilde{\mathbf{v}} = v_{n_1 \ldots n_m}$ of the polytope $P = \prod_{j=1}^{m} \Delta^{n_j}$ and $\bar{\lambda}$ is defined as in (5.2) satisfying

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$$\det A_{\mathbf{u}} = \begin{cases} -1 & \text{if } D(\mathbf{u}, \mathbf{u}_0) = \text{odd} \\ +1 & \text{if } D(\mathbf{u}, \mathbf{u}_0) = \text{even} \end{cases} \tag{5.6}$$

for $\mathbf{u} \in V(\bar{P})$. Then the matrix $A_{\tilde{\mathbf{v}}}$ can be characterized based on the determinant of the matrix.

Theorem 5.2 (Sarkar, Sau)

Let $X(\bar{P}, \bar{\lambda})$ be a (quasi)toric manifold where $\bar{P}$ is the vertex cut at $\tilde{\mathbf{v}} = v_{n_1 \dots n_m}$ of the polytope $P = \prod_{j=1}^{m} \Delta^{n_j}$ and $\bar{\lambda}$ is defined as in (5.2) satisfying

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Case 1: *If $\det A_{\tilde{\mathbf{v}}} = 0$, then*

$$\sum_{j=1}^{m} b_{N_j} = -1$$

and b_i can be arbitrary if $i \notin \{N_1, \dots, N_m\}$.

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$$b_i = \frac{(-1)^m}{\det A_{\tilde{\mathbf{v}}}} \sum_{q=1}^{n} A_{(i,q)}$$

for $i = 1, \ldots, n$ where $A_{(i,q)}$ is the (i,q)-th entry of the matrix $A_{\tilde{\mathbf{v}}}$.

A cohomology calculation following [4] leads us to

$$H^*(X(\bar{P}, \bar{\lambda})) \cong \mathbb{Z}[y_1, \ldots, y_m, y]/\overline{I} \tag{5.8}$$

where the homogeneous ideal $\overline{I}$ changes depending on the determinant of $A_{\tilde{\mathbf{v}}}$ while the generators remains same for all the cases.

[4] M. W. Davis, and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions,* Duke Math. J. **62**(1991).

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The elements $y_1, \ldots, y_m, y$ belong to $H^2(X(\bar{P}, \bar{\lambda}))$ and satisfy the following:

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2. $y^2 = (-1)^{m+1}(\det A_{\tilde{v}})yy_j$ *for any* $j = 1, \ldots, m$.

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A cohomology calculation following [4] leads us to

$$H^*(X(\bar{P}, \bar{\lambda})) \cong \mathbb{Z}[y_1, \ldots, y_m, y]/\bar{I} \tag{5.8}$$

where the homogeneous ideal $\bar{I}$ changes depending on the determinant of $A_{\tilde{\mathbf{v}}}$ while the generators remains same for all the cases.

Theorem 5.3 (Sarkar, Sau)

The elements $y_1, \ldots, y_m, y$ belong to $H^2(X(\bar{P}, \bar{\lambda}))$ and satisfy the following:

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For an element $z \in H^2(X(\bar{P}, \bar{\lambda}))$, the annihilator of z is defined by

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Since $\{\bar{F}, \bar{F}^j_{n_j}\}$ for $j = 1, \ldots, m$ are non-faces of $\bar{P}$, then $\mathrm{Ann}(cy)$ is of rank m for a nonzero constant c. The following lemma discusses about the converse.

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Let $P = \prod_{j=1}^{m} \Delta^{n_j}$ be a finite product of simplices as in (3.1) with $m \geqslant 2$ and $n \geqslant 3$ and $X(\bar{P}, \bar{\lambda})$ is a (quasi)toric manifold over the vertex cut at $\tilde{\mathbf{v}} = v_{n_1 \ldots n_m}$ of P.

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Theorem 5.6 (Sarkar, Sau)

Let $P = \prod_{j=1}^{m} \Delta^{n_j}$ be a product of simplices as in (3.1) with $m \geqslant 2$, $n \geqslant 3$ and $\bar{P}$ is a vertex cut of P along a vertex $\tilde{\mathbf{v}}$ such that $\det A_{\tilde{\mathbf{v}}} = (-1)^m$. Then $H^*(X(\bar{P}, \bar{\lambda}))$ and $H^*(X(\bar{P}, \bar{\lambda}'))$ are isomorphic as graded rings if and only if $H^*(X(P, \lambda))$ and $H^*(X(P, \lambda'))$ are isomorphic as graded rings.

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Thank You