On string quasitoric manifolds and their orbit polytopes

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Toric Topology 2022
March 23-25, Osaka City University

March 24, 2022

OUTLINE

1 Background
- Combinatorial pair (P, Λ)
- Quasitoric manifold $M(P, \Lambda)$
- String property

2 Main results
- Target and straightforward approach
- Low dimensional case
- Few facets case
- Real analogue

3 Further discussion

Definition (simple polytope and flag polytope)

A polytope is called *simple* if each codimension-k face is the intersection of exactly k facets.
A simple polytope is called *flag* if each family of pairwise intersecting facets has non-empty common intersection.

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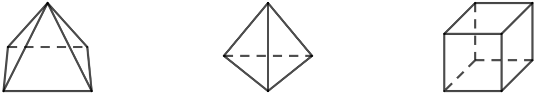

Definition (f-vector and h-vector)

Given an n-dimensional polytope P, let f_i denote the number of its i-dimensional faces and determine h_i by

$$\sum_{i=0}^{n} h_i s^{n-i} = \sum_{i=0}^{n} f_i (s-1)^i.$$

$\boldsymbol{f}(P) = (f_0, f_1, \ldots, f_{n-1}, 1)$ and $\boldsymbol{h}(P) = (h_0, h_1, \ldots, h_{n-1}, h_n)$ are called *f-vector* and *h-vector* of P respectively.

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- (Dehn-Sommerville relation)
  If P^n is simple, then $h_i = h_{n-i}$ for $0 \leq i \leq n$.

Definition (characteristic matrix)

For an n-dimensional simple polytope P with facets $\{F_i\}_{i=1}^m$, $\Lambda = (\boldsymbol{\lambda_1}, \cdots, \boldsymbol{\lambda_m}) \in Mat_{n\times m}(\mathbb{Z})$ is a corresponding *characteristic matrix* if the following nonsingular condition holds:

$$\forall\, p = \bigcap_{i=1}^{k} F_{j_i} \Rightarrow \det(\Lambda_p) = \det(\boldsymbol{\lambda_{j_1}}, \cdots, \boldsymbol{\lambda_{j_n}}) = \pm 1.$$

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Example

$P = C_2(3)$ i.e., triangle $\Rightarrow \Lambda = \begin{pmatrix} 1 & 0 & \delta_1 \\ 0 & 1 & \delta_2 \end{pmatrix}$ with $\delta_1, \delta_2 = \pm 1$.

Canonical Construction: $(P, \Lambda) \rightsquigarrow M(P, \Lambda)$

For each $p \in P$, there exists a unique face $f(p) = \bigcap_{i=1}^{k} F_{j_i}$ s.t. p is in the relative interior of $f(p)$.

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Regard $(\boldsymbol{\lambda}_{\boldsymbol{j_1}}, \cdots, \boldsymbol{\lambda}_{\boldsymbol{j_k}}) = (\lambda_{i,j}) \in Mat_{n\times k}(\mathbb{Z})$ as a map from T^k to T^n, sending $(t_1, \ldots, t_k)$ to $(\prod_{i=1}^k t_i^{\lambda_{1,i}}, \ldots, \prod_{i=1}^k t_i^{\lambda_{n,i}})$.
Let $G(p)$ denote the image.

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$M(P, \Lambda) = (P \times T^n)/\sim$ *with* $(p, \boldsymbol{t_1}) \sim (p, \boldsymbol{t_2}) \Leftrightarrow \boldsymbol{t_1}^{-1}\boldsymbol{t_2} \in G(p)$*.*

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Example

$P = \Delta^n \Rightarrow M(P, \Lambda) = \mathbb{C}P^n$.

Remark

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Example: n-dimensional dual cyclic polytope with more than $2^n - 1$ facets when $n \geq 4$.

2. Canonical Construction method can be applied to the case of moment-angle manifold.

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- Quotient of moment-angle manifold $\Rightarrow$ smooth, orientable and unitary structure on $M(P, \Lambda)$.
- Cell decomposition induced from star neighborhood of vertices $\Rightarrow$ explicit expression of $\beta^i(-), H_T^*(-), H^*(-)$ and $c_i(-), p_i(-)$.

Proposition (Davis-Januszkiewicz 1991)

The integral cohomology groups of $M(P, \Lambda)$ vanish in odd dimensions and therefore are free abelian in even dimensions. The Betti numbers are given by

$$\beta^{2i}(M(P, \Lambda)) = h_i(P) \qquad 0 \leq i \leq n$$

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- Dehn-Sommerville relation $\leftrightarrow$ Poincaré Duality.

Proposition (Davis-Januszkiewicz 1991)

Write $\Lambda = (\lambda_{i,j}) \in Mat_{n\times m}(\mathbb{Z})$, then the integral cohomology ring of $M(P,\Lambda)$ is given by

$$H^*(M(P,\Lambda)) \cong \mathbb{Z}[v_1,\dots,v_m]/(\mathcal{I}+\mathcal{J})$$

where face ring ideal $\mathcal{I}$ is generated by $\prod_{i=1}^{k} v_{j_i}$ for $\bigcap_{i=1}^{k} F_{j_i} = \emptyset$ and linear ideal $\mathcal{J}$ is generated by $\sum_{i=1}^{m} \lambda_{l,i}v_i$ for $1 \le l \le n$.

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- For each facet F_j, $M_j = \pi^{-1}(F_j)$ is called a characteristic submanifold, where $\pi : M(P,\Lambda) \to P$ is the natural projection. Generator v_j is the Poincaré dual of M_j.

Proposition (Davis-Januszkiewicz 1991)

Chern classes and Pontryagin classes are given by:

$$c(M(P,\Lambda)) = \prod_{j=1}^{m}(1+v_j) \qquad p(M(P,\Lambda)) = \prod_{j=1}^{m}(1+v_j^2).$$

In particular, $w_2(M(P,\Lambda)) \equiv c_1(M(P,\Lambda)) \equiv \sum_{j=1}^{m} v_j \pmod 2$ *and* $p_1(M(P,\Lambda)) = \sum_{j=1}^{m} v_j^2$.

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Example

$H^*(M(C_2(3),\Lambda)) \cong \mathbb{Z}[v]/\langle v^3\rangle$.
$w_2(M(C_2(3),\Lambda)) = (\delta_1+\delta_2+1)v = v$; $p_1(M(C_2(3),\Lambda)) = 3v^2$.

Definition (weakly equivariant homeomorphism)

Two T^n-manifolds M, N are *weakly equivariantly homeomorphic* (w.e.h.) if $\exists\ \phi \in \mathrm{Aut}T^n$ and $h \in \mathrm{Homeo}(M, N)$ s.t. $\forall\ t \in T^n$ and $m \in M,\ h(t \cdot m) = \phi(t) \cdot h(m)$.

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Three types of group actions on (P, Λ) inducing equivalence:
(1) column permutation by $\mathrm{Aut}(\partial P^*)$;
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Fact

There is a one-to-one correspondence between w.e.h. classes of quasitoric manifolds and equivalent classes of characteristic pairs.

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M is string $\Leftrightarrow w_1(M) = w_2(M) = p_1(M)/2 = 0$.

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- Natural target: topological properties expressed in a combinatorial way.
- Straightforward approach: express p_1 as a $\mathbb{Q}$-linear combination of certain $H^4(M(P,\Lambda))$ basis.
- w_2 and p_1 remain invariant under equivalence $\rightsquigarrow$ refined characteristic pair (P,Λ): $\cap_{i=1}^n F_i \neq \emptyset$ and $\Lambda = [\, \mathrm{I_n} \mid \Lambda_*]$.

Easy task: $w_2 = 0 \Leftrightarrow \sum_{i=1}^{n} \lambda_{i,j} = 1 \pmod 2$ for $1 \leq j \leq m$.

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- Independency of relations in $H^4(M(P,\Lambda))$:

$$\binom{m-n+1}{2} - \left[\binom{m}{2} - f_{n-2}(P)\right] = \beta^4(M(P,\Lambda)).$$

(Orlik-Raymond 1970) 4-dimensional quasitoric manifolds are homeomorphic to the equivariant connected sum of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$.

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- $C_2(m)$ can be realized as the orbit polytope of a string quasitoric manifold $\Leftrightarrow m \equiv 0 \pmod 2$.
- There exists exactly one homeomorphism class of string quasitoric manifold over $C_2(2m_0)$ for each $m_0 \geq 2$. While up to w.e.h., there are countably many equivalent classes.

Theorem A (S. 2022)

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- Parallel results are NOT valid in dimension > 3.

Example

$M(C_2(4) \times C_2(5), \Lambda)$ is string if Λ is equivalent to

$$\left(\begin{array}{cc|cc|cc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

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Proposition (S. 2022)

$C_2(m_1) \times C_2(m_2)$ can be realized as the orbit polytope of a string quasitoric manifold $\Leftrightarrow m_1, m_2 \geq 4$ and $m_1 m_2 \equiv 0 \pmod 2$.

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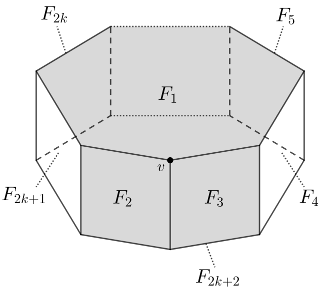

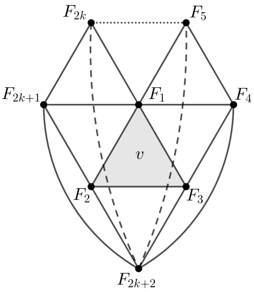

Example (string but NOT bundle type)

$$M(L_6,\Lambda) \text{ with } \Lambda = \left(\begin{array}{c|cc|cccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \end{array}\right).$$

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Theorem B (S. 2022)

If $M(L_{2k},\Lambda)$ is string, then there exist $\{M(L_{2k_i},\Lambda_i)\}_{i=1}^{s}$ s.t.
(1) $M(L_{2k_i},\Lambda_i)$ is of bundle type and string for $1 \le i \le s$;
(2) $M(L_{2k},\Lambda) = M(L_{2k_1},\Lambda_1)\widetilde{\#^e}\cdots\widetilde{\#^e}M(L_{2k_s},\Lambda_s)$ up to w.e.h..

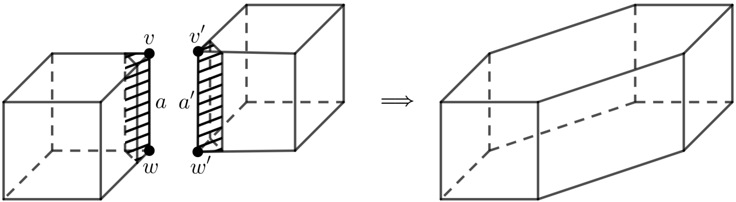

Figure: $L_6 = L_4 \#^e L_4$

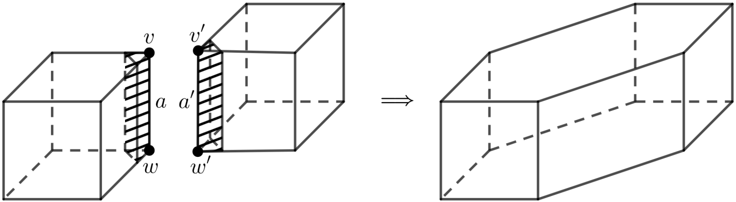

Figure: $L_6 = L_4 \#^e L_4$

Edge connected sum $\#^e$ together with compatible coloring $\rightsquigarrow$ equivariant edge connected sum $\widetilde{\#^e}$.

Example (decomposition via $\widetilde{\#}^e$)

$$\Lambda = \left(\begin{array}{c|cc|cccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \end{array}\right)$$

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$M(L_6, \Lambda) = M(L_4, \Lambda_1)\widetilde{\#}^e M(L_4, \Lambda_2)$ with Λ_1, Λ_2 equivalent to

$$\left(\begin{array}{c|cc|cc|c} 1 & 0 & 0 & 2 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right).$$

Key Observation

Given an n-dimensional simple polytope P, if there exist facets F and $\{F_{j_i}\}_{i=1}^{n}$ s.t. $\bigcap_{i=1}^{n} F_{j_i} \neq \emptyset$ and $F \cap F_{j_i} \neq \emptyset$ for $1 \leq i \leq n$, then P can NOT be realized as the orbit polytope of a string quasitoric manifold.

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Example

1. $P = \prod_{i=1}^k P_i$ with some P_i^* 2-neighborly;
2. P with a triangular 2-face.

Proposition (Blind-Blind 1992)

If an n-dimensional simple polytope P is triangle-free, then the number of facets $f_{n-1}(P) \geq 2n$. Moreover,
(1) $f_{n-1}(P) = 2n \Rightarrow P = I^n$;
(2) $f_{n-1}(P) = 2n + 1 \Rightarrow P = C_2(5) \times I^{n-2}$;
(3) $f_{n-1}(P) = 2n + 2 \Rightarrow P = C_2(6) \times I^{n-2}$ *or* $Q \times I^{n-3}$ *or* $C_2(5) \times C_2(5) \times I^{n-4}$ *where Q is an edge cut of* $C_2(5) \times I$.

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(3) $f_{n-1}(P) = 2n+2 \Rightarrow P = C_2(6) \times I^{n-2}$ or $Q \times I^{n-3}$ or $C_2(5) \times C_2(5) \times I^{n-4}$ where Q is an edge cut of $C_2(5) \times I$.

- label facets and choose basis appropriately $\rightsquigarrow$ explicit formula for $p_1 = \sum c_{i,j} v_i v_j$.

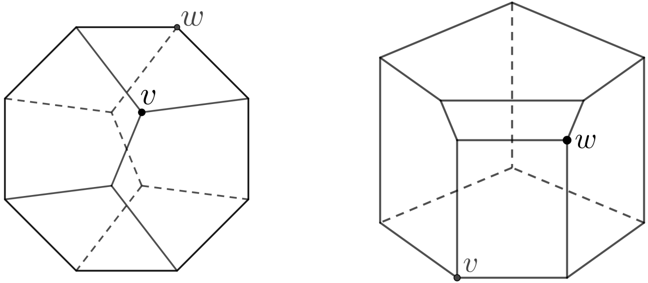

Figure: Q as an edge cut of $C_2(5) \times I$

Label the facets of I^n s.t. $F_i \cap F_{n+i} = \emptyset$ for $1 \leq i \leq n$. Choose the basis of $H^4(M(I^n, \Lambda))$ as $\{v_i v_j\}_{n+1 \leq i < j \leq 2n}$.

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Notation

Given $\Lambda = (\lambda_{s,t})_{n \times m}$, write $\rho_j = \sum_{i=1}^{n} \lambda_{i,j}^2 + 1$ for $1 \leq j \leq m$ and $\rho_{j_1,j_2} = \rho_{j_2,j_1} = 2\sum_{i=1}^{n} \lambda_{i,j_1}\lambda_{i,j_2}$ for $1 \leq j_1 < j_2 \leq m$.

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Proposition (S. 2022)

$M(I^n, \Lambda)$ is string $\Leftrightarrow \sum_{k=1}^n \lambda_{k,i} \equiv 1 \pmod 2$ for $n+1 \leq i \leq 2n$ and $\lambda_{i-n,i}\lambda_{i-n,j}\rho_i + \lambda_{j-n,j}\lambda_{j-n,i}\rho_j = \rho_{i,j}$ for $n+1 \leq i < j \leq 2n$.

Definition (Bott manifold)

Given a $\mathbb{C}P^1$-bundle tower:

$$B^{2n} \xrightarrow{\mathbb{C}P^1} B^{2n-2} \xrightarrow{\mathbb{C}P^1} \cdots \xrightarrow{\mathbb{C}P^1} B^2 \xrightarrow{\mathbb{C}P^1} pt,$$

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- Bott manifold may not be spin or string.

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- More generalized version: If $M(P \times I^n, \Lambda)$ is string, then $M'(I^n, \Lambda')$ is a Bott manifold, where Λ' is the restricted characteristic matrix.

  Caution: $M'(I^n, \Lambda')$ is not string in general.

Theorem D (S. 2022)

$M(I^n \# P^n, \Lambda)$ is string iff it is w.e.h. to $M(I^n, \Lambda_L) \widetilde{\#} M(P^n, \Lambda_R)$ with both $M(I^n, \Lambda_L)$ and $M(P^n, \Lambda_R)$ string.

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- Connected sum $\# \rightsquigarrow$ equivariant connected sum $\widetilde{\#}$.
- "If" part follows from definition and "only if" part follows from decomposition guaranteed by string property.

Proposition (Dobrinskaya 2001)

Suppose $A \in Mat_{n\times n}(\mathbb{Z})$ *and every proper principal minor of* A *is equal to 1. If* $\det A = 1$*, then* A *is conjugate to a unipotent upper triangular matrix. If* $\det A = -1$*, then* A *is conjugate to*

$$\begin{pmatrix} 1 & b_1 & 0 & \cdots & 0 \\ 0 & 1 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-1} \\ b_n & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $\prod_{i=1}^{n} b_i = (-1)^n \cdot 2$.

Label the facets of $C_2(5) \times I^{n-2}$ s.t. $\{F_i\}_{i=1}^{5}$ correspond to five facets of $C_2(5)$ and $F_i \cap F_{n-2+i} = \emptyset$ for $6 \leq i \leq n+3$.
Choose the basis of $H^4(M(P,\Lambda))$ as v_4v_5, $\{v_iv_j\}_{n+4\leq i<j\leq 2n+1}$, $\{v_iv_j\}_{3\leq i\leq 5, n+4\leq j\leq 2n+1}$ and write coefficients as $c_{i,j}, c'_{i,j}$ and $c_{4,5}$ respectively.

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Notation

Given $\Lambda = (\lambda_{s,t})_{n \times m}$, write $\Delta_{j_1,j_2} = \det \begin{pmatrix} \lambda_{1,j_1} & \lambda_{1,j_2} \\ \lambda_{2,j_1} & \lambda_{2,j_2} \end{pmatrix}$ for $1 \leq j_1 < j_2 \leq m$ and $l_j = \Delta_{j-1,j} \cdot \Delta_{j,j+1} \cdot \Delta_{j+1,j-1}$ for $1 \leq j \leq 5$ with subscripts taken modulo 5.

- $c_{i,j} = -\lambda_{i-n-1,j}\rho_i - \lambda_{j-n-1,i}\rho_j + \rho_{i,j}$;
  $c'_{3,j} = -\lambda_{1,j}\rho_3 - \lambda_{j-n-1,3}\rho_j + \rho_{3,j}$;
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Replace T^n by $\mathbb{Z}_2^n$ in Canonical Construction: real characteristic pair $(P, \Lambda) \rightsquigarrow$ small cover $M_{\mathbb{R}}(P, \Lambda)$.

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- $H^*(M_{\mathbb{R}}(P, \Lambda); \mathbb{Z}_2) \cong \mathbb{Z}_2[v_1, \ldots, v_m]/(\mathcal{I} + \mathcal{J})$.
- $w(M_{\mathbb{R}}(P, \Lambda)) = \prod_{i=1}^m (1 + v_i)$; $p(M_{\mathbb{R}}(P, \Lambda)) = 1$.

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- Real analogue: different results via simpler arguments.
- Partial results in certain cases can be found in the work of Choi-Masuda-Oum, Dsouza-Uma and Huang.

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- $\prod_{i=1}^{k} \Delta^{n_i}$ with $n_i \geq 2$ can be realized as the orbit polytope of a string small cover $\Leftrightarrow n_i \equiv 1 \pmod 2$ for $1 \leq i \leq k$ and $\exists\, i_0$ s.t. $n_{i_0} \equiv 3 \pmod 4$.

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In particular, if P^n can be realized as the orbit polytope of a string quasitoric manifold, is there an upper bound for the chromatic number $\gamma(P^n)$?

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Search for nontrivial string manifolds with the help of $M(P, \Lambda)$
String bordism & Stolz Conjecture