

# A Chevalley formula for the Factorial $GP$ functions

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# Contents

- 1 Definition of the Factorial  $GP$  Functions
- 2 Factorial  $GP$  functions and orthogonal Grassmannian variety
- 3 Chevalley formula of Factorial  $GP$  functions

# Factorial $GP$ Functions

Factorial  $GP$  functions are special functions that represent the Schubert classes in torus equivariant  $K$  theory on maximal orthogonal Grassmannian

# Strict Partitions and Shifted Young Diagrams

## Definition Strict Partition

$\lambda = (\lambda_1, \dots, \lambda_\ell)$  is strict partition: strictly decreasing sequence of positive integers .

$\ell$  is length of  $\lambda$ .

$\emptyset$  is a strict partition of length 0.

$\mathcal{SP}_n$ : a set of strict partittions of length  $\leq n$ .

$\mathcal{SP}(n)$ : a set of strict partitions such that the first entry  $\leq n$ .

## example

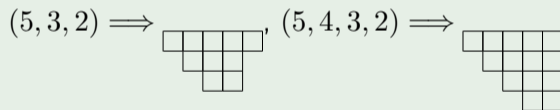
$(4, 2), (5, 3, 2), (5, 4, 3), (6) \in \mathcal{SP}_3$ .

## Definition Shifted Young Diagram

Let  $\lambda$  be a strict partition.

$\mathbb{D}(\lambda) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} | i \leq j \leq \lambda + i - 1, 1 \leq \ell(\lambda)\}$ : shifted Young diagram of  $\lambda$

## Example



We identify a strict partition and its shifted Young diagram.

## Definition   Inclusion of strict partitions

Let  $\lambda, \mu$  be strict partitions.

## Example

$$(2, 1) \subset (3, 2, 1), (1) \subset (2), (3, 1) \subset (3, 2), \dots$$

# Factorial $GP$ functions (tableaux representation)

## Factorial $GP$ functions (Ikeda–Naruse)

$$\lambda \in \mathcal{SP}_n$$

$$GP_{\lambda}(x_1, \dots, x_n | b) = \sum_{T \in \mathcal{T}(\lambda)} \beta^{|T| - |\lambda|} (x | b)^T$$

# Set Valued Tableaux ( $\mathcal{T}(\lambda)$ )

## Definition (Set-Valued Tableaux)

$\lambda \in \mathcal{SP}_n$ .  $\mathcal{A} := \{1' < 1 < 2' < 2 < \dots < n' < n\}$ : ordered alphabet. A set-valued tableau  $T$  of shape  $\lambda$  is an assignment  $T : \mathbb{D}(\lambda) \rightarrow 2^{\mathcal{A}}$

- (1)  $\max T(i, j) \leq \min T(i, j+1), \max T(i, j) \leq \min T(i+1, j),$
- (2) Each  $a$  (non prime number) appears at most once in each column,
- (3) Each  $a'$  (prime number) appears at most once in each row,
- (4) If  $i$  is odd number, then  $T(i, i) \subset \{1, 2', 3, 4' \dots\}$  and if  $i$  is even number  $T(i, i) \subset \{1', 2, 3', 4, \dots\}.$

We denote by  $\mathcal{T}(\lambda)$  the set of all set-valued tableaux of shape  $\lambda$ .

## Example

If  $n = 2$ ,  $\lambda = (2, 1)$ , then  $\mathcal{T}(\lambda)$  consist of the following three tableaux.

1	1
	2

1	2'
	2

1	1, 2'
	2

These tableaux are not elements of  $\mathcal{T}(\lambda)$ .

1'	1
	2

1	1
	2'

1	2
	2

$$(x|b)^T$$

$$x \oplus y = x + y + \beta xy, \quad x \ominus y = \frac{x - y}{1 + \beta y}.$$

For each  $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$  and  $a \in \mathcal{A}$ , we define

$$w(i, j; a) := \begin{cases} x_a \oplus b_{-i+j+1} & a = 1, 2, \dots, n \\ x_{|a|} \ominus b_{-i+j+1} & a = 1', 2', \dots, n' \end{cases}.$$

For  $T \in \mathcal{T}(\lambda)$ , we define

$$(x|b)^T = \prod_{(i,j) \in \lambda, a \in T(i,j)} w(i, j; a)$$

# Factorial $GP$ functions (tableaux representation)

## Factorial $GP$ function(Ikeda–Naruse)

For  $\lambda \in \mathcal{SP}_n$

$$GP_{\lambda}(x_1, \dots, x_n | b_1, b_2, \dots) = \sum_{T \in \mathcal{T}(\lambda)} \beta^{|T| - |\lambda|} (x|b)^T$$

## Example

Let  $n = 2$  and  $\lambda = (2, 1)$ , then  $\mathcal{T}(\lambda) = \{T_1, T_2, T_3\}$ .

$$T_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & 2 \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|} \hline 1 & 2' \\ \hline & 2 \\ \hline \end{array} \quad T_3 = \begin{array}{|c|c|} \hline 1 & 1, 2' \\ \hline & 2 \\ \hline \end{array}$$

$$(x|b)^{T_1} = (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_1)$$

$$(x|b)^{T_2} = (x_1 \oplus b_1)(x_2 \ominus b_2)(x_2 \oplus b_1)$$

$$(x|b)^{T_3} = (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \ominus b_2)(x_2 \oplus b_1)$$

## Example

$$\begin{aligned}
GP_{\boxplus}(x_1, x_2 | b_1, b_2, \dots) &= \sum_{T \in \mathcal{T}(2,1)} \beta^{|T| - |(2,1)|} (x|b)^T \\
&= (x|b)^{T_1} + (x|b)^{T_2} + \beta (x|b)^{T_3} \\
&= (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_1) + (x_1 \oplus b_1)(x_2 \ominus b_2)(x_2 \oplus b_1) \\
&\quad + \beta (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \ominus b_2)(x_2 \oplus b_1) \\
&= (x_1 \oplus b_1)(x_2 \oplus b_1) \{ (x_1 \oplus b_2) + (x_2 \ominus b_2) + \beta (x_1 \oplus b_2)(x_2 \ominus b_2) \} \\
&= (x_1 \oplus b_1)(x_2 \oplus b_1)((x_1 \oplus b_2) \oplus (x_2 \ominus b_2)) \\
&= (x_1 \oplus b_1)(x_2 \oplus b_1)(x_1 \oplus x_2)
\end{aligned}$$

## Factorial $GP$ functions (Hall-Littlewood-type formula)

$$x \oplus y = x + y + \beta xy, \quad x \ominus y = \frac{x - y}{1 + \beta y}.$$

For  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \mathcal{SP}_n$ , we set

$$[x|b]^\lambda = \prod_{j=1}^{\lambda_1} (x_1 \oplus b_j) \prod_{j=1}^{\lambda_2} (x_2 \oplus b_j) \cdots \prod_{j=1}^{\lambda_\ell} (x_\ell \oplus b_j).$$

### Factorial $GP$ function(Ikeda–Naruse)

$$GP_\lambda(x_1, \dots, x_n | b_1, b_2, \dots) = \frac{1}{(n - \ell)!} \sum_{\sigma \in S_n} \sigma \left[ [x|b]^\lambda \prod_{1 \leq i < j \leq n, i \leq \ell} \frac{x_i \oplus x_j}{x_i \ominus x_j} \right].$$

where  $\sigma \in S_n$  acts on the variables  $x_1, \dots, x_n$ .

## Example

$$\begin{aligned}
GP_{\boxplus}(x_1, x_2 | b_1, b_2, \dots) &= \frac{1}{(2-2)!} \sum_{\sigma \in S_2} \sigma \left[ (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_1) \frac{x_1 \oplus x_2}{x_1 \ominus x_2} \right] \\
&= (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_1) \frac{x_1 \oplus x_2}{x_1 \ominus x_2} \\
&\quad + (x_2 \oplus b_1)(x_2 \oplus b_2)(x_1 \oplus b_1) \frac{x_2 \oplus x_1}{x_2 \ominus x_1} \\
&= (x_1 \oplus b_1)(x_2 \oplus b_1)(x_1 \oplus x_2) \left\{ \frac{(x_1 \oplus b_2) \ominus (x_2 \oplus b_2)}{x_1 \ominus x_2} \right\} \\
&= (x_1 \oplus b_1)(x_2 \oplus b_1)(x_1 \oplus x_2)
\end{aligned}$$

# Properties of Factorial $GP$ Functions

- $GP_\lambda(x_1, \dots, x_n | b)$  is a symmetric polynomial  $x_1, \dots, x_n$ .
- The coefficients of  $GP_\lambda(x_1, \dots, x_n | b)$  as a polynomial are non negative integers.
- $GP_\lambda(x_1, \dots, x_n | b)$  is a homogeneous polynomial of degree  $|\lambda|$ .  
 $(\deg(x_i) = \deg(b_i) = 1, \deg(\beta) = -1)$

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# Maximal orthogonal Grassmannian variety and Schubert variety

$\{e_{n+1}^*, \dots, e_1^*, e_1, \dots, e_{n+1}\}$ : orderd basis of  $\mathbb{C}^{2n+2}$

$(-, -)$ : symmetric bilinear form which satisfies

$$(e_i, e_j) = (e_i^*, e_j^*) = 0, (e_i^*, e_j) = \delta_{i,j}$$

$X = OG(n+1, 2n+2) = \{V \subset \mathbb{C}^{2n+2} \mid \dim(V) = n+1, (V, V) = 0\}$ : maximal orthogonal Grassmannian variety

$$F_i = \langle e_{n+1}^*, \dots, e_{n+2-i}^* \rangle$$

$F_\bullet : 0 \subset F_1 \subset F_2 \subset \dots \subset F_{n+1} \subset \mathbb{C}^{2n+2} \mid \dim(F_i) = i, (F_i, F_i) = 0$  (called isotropic flag).

$\Omega_\lambda(F_\bullet) = \{V \in X \mid \dim(V \cap F_{n+1-\lambda_i}) \geq i, 1 \leq i \leq \ell(\lambda)\}$ : Schubert variety

# Torus Equivariant K-theory of Maximal Orthogonal Grassmannian Variety

$K_T(X)$ : Grothendieck group of the abelian category of torus equivariant coherent sheaves on  $X$

$\mathcal{O}_{\Omega_\lambda}$ : structure sheaf of  $\Omega_\lambda$

$\mathcal{O}_\lambda = [\mathcal{O}_{\Omega_\lambda}] \in K_T(X)$ : Schubert class.  $\{\mathcal{O}_\lambda | \lambda \in \mathcal{SP}\}$  form a basis in  $K_T(X)$  as  $K_T(pt)$  module.

## The Fundamental Problem of Schubert Calculus

$$\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_{\nu} c_{\lambda, \mu}^{\nu} \mathcal{O}_\nu$$

Compute  $c_{\lambda, \mu}^{\nu}$ .

# Schubert classes and Factorial $GP$ functions

## Theorem(Ikeda–Naruse)

There exists a surjective homomorphism.

$$\begin{aligned} \pi_n : R(T) \otimes_{\mathbb{Z}[\beta]} G\Gamma_n &\longrightarrow K_T(X) \\ GD_\lambda(x|1 - e^{t_1}, \dots, 1 - e^{t_{n+1}}, 0, \dots) &\mapsto \begin{cases} \mathcal{O}_\lambda & \lambda \in \mathcal{SP}(n) \\ 0 & \lambda \notin \mathcal{SP}(n) \end{cases} \end{aligned}$$

Where  $GD_\lambda(x|b) = \begin{cases} GP_\lambda(x_1, \dots, x_n|b_1, b_2, \dots) & n: \text{even number} \\ GP_\lambda(x_1, \dots, x_n, 0|b_1, b_2, \dots) & n: \text{odd number} \end{cases}$

$\mathcal{R}(T) = \mathbb{Z}[e^{\pm t_1}, \dots, e^{\pm t_{n+1}}], G\Gamma_n = \mathbb{Z}[\beta][GP_\lambda(x_1, \dots, x_n)|\lambda \in \mathcal{SP}_n]$ .

# Structure constants of non factorial $GP$ functions ,factorial Schur $P$ functions and Schur $P$ Functions

$$X = OG(n+1, 2n+2) = \{V \subset \mathbb{C}^{2n+2} \mid \dim(V) = n+1, (V, V) = 0\}$$

$$\begin{array}{ccc}
 GP_{\lambda}(x|b) & \xrightarrow{\forall b_i=0} & GP_{\lambda}(x) \\
 \beta=0 \downarrow & & \beta=0 \downarrow \\
 P_{\lambda}(x|b) & \xrightarrow{\forall b_i=0} & P_{\lambda}(x)
 \end{array}
 \qquad
 \begin{array}{ccc}
 K_T(X) & \xrightarrow{\forall b_i=0} & K(X) \\
 \beta=0 \downarrow & & \beta=0 \downarrow \\
 H_T^*(X) & \xrightarrow{\forall b_i=0} & H^*(X)
 \end{array}$$

Schur  $P$  functions  $\implies$  Littlewood–Richardson rule (Stembridge, 1989)

Factorial  $P$  functions  $\implies$  Pieri rule (Cho–Ikeda, 2011)

Non factorial  $GP$  functions  $\implies$  Littlewood–Richardson rule (Clifford–Thomas–Yong, 2014)

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- 1 Definition of the Factorial  $GP$  Functions
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$$GP_{\lambda}(x|b) \cdot GP_{\mu}(x|b) = \sum_{\nu} c_{\lambda,\mu}^{\nu} GP_{\nu}(x|b)$$

We want to calculate the coefficients  $c_{\lambda,\mu}^{\nu}$ .

The case of  $\mu = (1, 0, 0, \dots) = \square$

$\implies$  Chevalley formula (2018, Buch–Chaput–Mihalcea–Perrin)

The case of  $\mu = (\mu_1, 0, 0, \dots)$

$\implies$  Pieri formula is an open problem.

## Definition Rook Strip

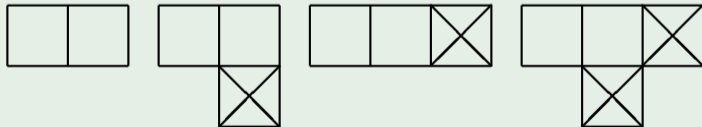
For  $\lambda \subset \mu$  ( $\lambda_i \leq \mu_i, \forall i$ )

$\mu/\lambda$ : collection of boxes that belong to the diagram of  $\mu$  but not to that of  $\lambda$

$\mu/\lambda$  is rook strip.  $\stackrel{\text{def}}{\iff} \mu/\lambda$  has at most one box in all rows and all columns.

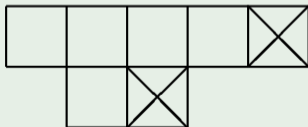
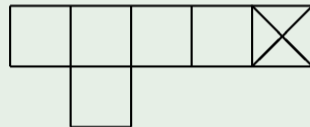
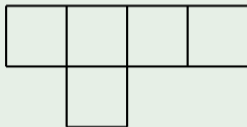
## Example

$\lambda = (2), \mu/\lambda$ :rook strip  $\implies \mu = (2), (2, 1), (3), (3, 1)$



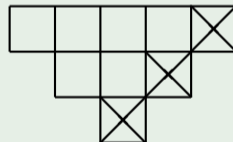
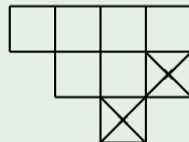
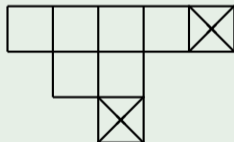
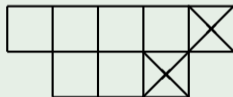
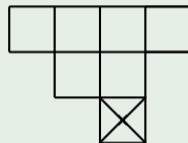
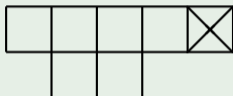
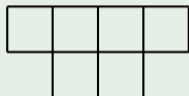
## Example

$\lambda = (4, 1)$ ,  $\mu/\lambda$ : rook strip



## example

$\lambda = (4, 2), \mu/\lambda: \text{rook strip} \implies \mu = (4, 2), (5, 2), (4, 3), (4, 2, 1), (5, 3), (5, 2, 1), (4, 3, 1), (5, 3, 1)$



## Chevalley Formula (Buch–Chaput–Mihalcea–Perrin)

$n$  : even number,  $\lambda \in \mathcal{SP}_n$ ,  $\ell$  : length of  $\lambda$ ,  $r.s$ : rook strip.

$$GP_\lambda(x|b) \cdot GP_\square(x|b) = \beta^{-1} \{ \Pi(b_\lambda) - 1 \} GP_\lambda(x|b) + \Pi(b_\lambda) \sum_{\substack{\mu \in \mathcal{SP}_n \\ \mu/\lambda: r.s, \\ \mu \neq \lambda}} \beta^{|\mu/\lambda|-1} GP_\mu(x|b)$$

$$\left( = \Pi(b_\lambda) \sum_{\substack{\mu \in \mathcal{SP}_n \\ \mu/\lambda: r.s}} \beta^{|\mu/\lambda|-1} GP_\mu(x|b) - \beta^{-1} GP_\lambda(x|b) \right)$$

$$\Pi(b_\lambda) = \begin{cases} \prod_{i=1}^{\ell} (1 + \beta b_{\lambda_i+1})^{-1} & \ell : \text{even} \\ (1 + \beta b_1)^{-1} \prod_{i=1}^{\ell} (1 + \beta b_{\lambda_i+1})^{-1} & \ell : \text{odd} \end{cases}.$$

## Example

$\lambda = (2), \mu/\lambda: \text{rook strip} \implies \mu = (2), (2, 1), (3), (3, 1)$

$$GP_{\square\square}(x|b) \cdot GP_{\square}(x|b) = \beta^{-1} \{ \Pi(b_{\square\square}) - 1 \} GP_{\square\square}(x|b) \\ + \Pi(b_{\square\square}) \{ GP_{\square\square\square}(x|b) + GP_{\square\square\square}(x|b) + \beta GP_{\square\square\square}(x|b) \}$$

Where

$$\Pi(b_{\square\square}) = (1 + \beta b_3)^{-1} (1 + \beta b_1)^{-1} .$$

## Example

$\lambda = (4, 1), \mu/\lambda$ :rook strip  $\mu = (4, 1), (4, 2), (5, 1), (5, 2)$

$$\begin{aligned}
 GP_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}}(x|b) \cdot GP_{\square}(x|b) &= \beta^{-1} \{ \Pi(b_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}}) - 1 \} GP_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}}(x|b) \\
 &+ \Pi(b_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}}) \{ GP_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}}(x|b) + GP_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \end{array}}(x|b) + \beta GP_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \end{array}}(x|b) \}
 \end{aligned}$$

Where

$$\Pi(b_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}}) = (1 + \beta b_5)^{-1} (1 + \beta b_2)^{-1} .$$

## Example

$\lambda = (4, 2)$ ,  $\mu/\lambda$ : rook strip

$\implies \mu = (4, 2), (5, 2), (4, 3), (4, 2, 1), (5, 3), (5, 2, 1), (4, 3, 1), (5, 3, 1)$

If  $x = (x_1, x_2)$ , then

$$\begin{aligned}
 & GP_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}(x|b) \cdot GP_{\square}(x|b) \\
 = & \beta^{-1} \{ \Pi(b_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}) - 1 \} GP_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}(x|b) \\
 + & \Pi(b_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}) \left\{ GP_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}(x|b) + GP_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}(x|b) + GP_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}(x|b) + \beta GP_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}(x|b) \right\}
 \end{aligned}$$

Where

$$\Pi(b_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}) = (1 + \beta b_5)^{-1} (1 + \beta b_3)^{-1} .$$

# Proof

Define coefficients  $c_\lambda^\mu = c_\lambda^\mu(\beta, b)$

$$\frac{GP_\lambda(x|b)(1 + \beta GP_\square(x|b))}{\Pi(b_\lambda)} = \sum_{\mu \in \mathcal{SP}_n} \beta^{|\mu| - |\lambda|} c_\lambda^\mu GP_\mu(x|b)$$

- 1  $c_\lambda^\mu$  is a rational functions of  $\beta b_1, \beta b_2, \dots$  .
- 2  $c_\lambda^\mu$  is a polynomial  $\beta, b_1, b_2, \dots$  .
- 3  $\deg_\beta c_\lambda^\mu \leq 0$  .

# Proof

By claim 1,2, and 3,  $c_\lambda^\mu$  is an integer.  $\implies c_\lambda^\mu(\beta, b) = c_\lambda^\mu(\beta, 0)$

Use the Chevalley formula of "NON" factorial  $GP$  functions (Buch–Ravikumar 2014, Clifford–Thomas–Yong, 2014).

Thank you for your time and attention.