Relative version of fillable complex and Whitehead products in polyhedral products

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Polyhedral products

- K : an abstract simplicial complex with vertex set $[m] = \{1, 2, ...m\}$
- $(\underline{X}, \underline{A}) := \{(X_i, A_i)\}_{i=1}^m$: a collection of based CW-pairs
- The polyhedral product of $(\underline{X}, \underline{A})$ associated with K defined as

$$\mathcal{Z}(K; (\underline{X}, \underline{A})) := \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subset \prod_{i=1}^m X_i$$

where

$$(\underline{X}, \underline{A})^\sigma := \prod_{i=1}^m Y_i^\sigma$$

such that

$$Y_i^\sigma := X_i \text{ for } i \in \sigma,\ Y_i^\sigma := A_i \text{ for } i \notin \sigma.$$

Polyhedral products

Example

- Let $K_0 = \{1, 2\}$, then

$$\mathcal{Z}(K_0; (\underline{D^2}, \underline{S^1})) = (D^2 \times S^1) \cup (S^1 \times D^2) = S^2.$$

$$\mathcal{Z}(K_0; (\underline{S^2}, \underline{*})) = (S^2 \times *) \cup (* \times S^2) = S^2 \vee S^2.$$

- If $K = L_0 * L_1 := \{\sigma \sqcup \tau \mid \sigma \in L_0,\ \tau \in L_1\}$, then

$$\mathcal{Z}(K; (\underline{D^2}, \underline{S^1})) = \mathcal{Z}(L_0; (\underline{D^2}, \underline{S^1})) \times \mathcal{Z}(L_1; (\underline{D^2}, \underline{S^1})).$$

Problem

- $\mathcal{Z}_K(\underline{X}) := \mathcal{Z}(K; (C\underline{X}, \underline{X}))$, $\mathrm{DJ}_K(\underline{X}) := \mathcal{Z}(K; (\underline{X}, \underline{*}))$.
- $(C\underline{X}, \underline{X}) \twoheadrightarrow (\Sigma\,\underline{X}, \underline{*})$ induces the natural map w_K

$$w_K : \mathcal{Z}_K(\underline{X}) \twoheadrightarrow \mathrm{DJ}_K(\Sigma\,\underline{X}).$$

- For example, when $K = \{1, 2\}$ and $\underline{X} = S^1$,

$$\mathcal{Z}_K(\underline{S^1}) = S^3 \xrightarrow{w_K} S^2 \vee S^2 = \mathrm{DJ}_K(\underline{S^2})$$

  is Whitehead product.

$\rightsquigarrow$ For which simplicial complex K is the map w_K described by Whitehead product ?

Totally fillable complex

- A set σ is a minimal non-face of K if $\sigma \notin K$ and $\partial\sigma \subset K$.

K.Iriye and D.Kishimoto defined "fillable complexes".

- K is fillable

  $\Leftrightarrow$ There exist a set of minimal non-faces $\{\sigma_1, \dots, \sigma_r\} =: \mathcal{F}(K)$

  such that $|K \cup \sigma_1 \cup \dots \cup \sigma_r|$ is contractible.

- K is totally fillable $\Leftrightarrow$ K_I is fillable for any $I \subset [m]$.

Whitehead products in polyhedral products

Theorem [Iriye-Kishimoto 2]

Suppose that $\underline{X} = \Sigma\,\underline{Y}$, and K is totally fillable. Then,

- $\mathcal{W}_K(\underline{X}) := \bigvee_{\emptyset \neq I \subset [m]} \{(\bigvee_{\tau \in \mathcal{F}(K_I)} S^{|\tau|-1}) \wedge \underline{X}^{\wedge I}\} \simeq \mathcal{Z}_K(\underline{X})$ and
- $S^{|\sigma|-1} \wedge \underline{X}^{\wedge I} \hookrightarrow \mathcal{W}_K(\underline{X}) \xrightarrow{\simeq} \mathcal{Z}_K(\underline{X}) \xrightarrow{w_K} \mathrm{DJ}_K(\Sigma\,\underline{X})$

  is an iterated Whitehead product $[[\cdots[w_{\partial\sigma}, a_{i_1}], \cdots], a_{i_k}]]$,

  where $i_1 < \cdots < i_k$ is a ordering of $I \setminus \sigma$ and

  $a_{i_k} : \Sigma\, X_{i_k} \hookrightarrow \mathrm{DJ}_K(\Sigma\,\underline{X})$.

- $w_{\partial\sigma}$ is a higher Whitehead product.

Totally fillable complex relative to $\mathcal{L} = \{L(I)\}$

- Let L be a subcomplex of K,

  and $\mathcal{L} = \{L(I)\}$ be a system of subcomplexes of K.

- <u>Definition</u> (K, L) is fillable if

  there exist a set of minimal non-faces $\{\sigma_1, \ldots, \sigma_r\} =: \mathcal{F}(K)$

  such that $(|K \cup \{\sigma_1, \sigma_2, ..., \sigma_r\}|, |L|)$ is a DR-pair.

- <u>Definition</u> K is totally fillable relative to $\mathcal{L}$ if

  $(K_I, L(I))$ is fillable and $\varphi_{L(I)} \simeq *$ for all $I \subset [m]$.

- The maps $\varphi_{L(I)}$ are defined later in the slide.

Whitehead products in polyhedral products

Main theorem

If $\underline{X} = \Sigma\,\underline{Y}$, and K is totally fillable relative to $\mathcal{L} = \{L(I)\}$, then

- $\mathcal{W}_K(\underline{X}) := \bigvee_{\emptyset \neq I \subset [m]} \{(\Sigma\,L(I) \vee \bigvee_{\tau \in \mathcal{F}(K_I)} S^{|\tau|-1}) \wedge \underline{X}^{\wedge I}\} \simeq \mathcal{Z}_K(\underline{X})$ and
- $|\Sigma\,L(I)| \wedge \underline{X}^{\wedge I} \hookrightarrow \mathcal{W}_K(\underline{X}) \xrightarrow{\simeq} \mathcal{Z}_K(\underline{X}) \xrightarrow{w_K} \mathcal{D}J_K(\Sigma\,\underline{X})$

  is the iterated Whitehead product $[[\cdots[w_{L(I)}, a_{i_1}], \cdots], a_{i_k}]$, where

  $i_1 < \cdots < i_k$ is a ordering of $I \setminus \mathrm{vert}(L(I))$, and

  $a_{i_k} : \Sigma\,X_{i_k} \hookrightarrow \mathcal{D}J_K(\Sigma\,\underline{X})$.

- $w_{L(I)}$ may be not a higher Whitehead product.

Cone decomposition of $\mathcal{Z}_K(\underline{X})$ by Iriye-Kishimoto

- a real moment-angle complex $\mathbb{R}\mathcal{Z}_K := \mathcal{Z}_K(\underline{\{0,1\}})$.
- ${\mathbb{R}\mathcal{Z}_K}^i := \mathbb{R}\mathcal{Z}_K \cap \prod_m^i \{\underline{0,1}\}$,

  where $\prod_m^i \{\underline{0,1}\}$ is the i-th fat wedge of $\{\underline{0,1}\}$ with a base point 0.
- $\varphi_K : |Sd(K)| = |K| \to {\mathbb{R}\mathcal{Z}_K}^{|K|-1}$ to satiyfy

$$\varphi_K(\sigma) = x^\sigma \iff (x^\sigma)_i = \begin{cases} 0, & i \in \sigma \\ 1, & i \notin \sigma. \end{cases}$$

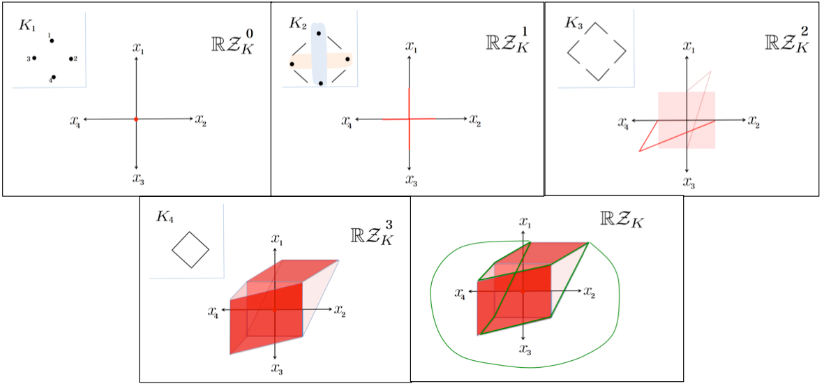
- Theorem[Iriye-Kishimoto 1]

$${\mathbb{R}\mathcal{Z}_K}^i = {\mathbb{R}\mathcal{Z}_K}^{i-1} \cup_{\coprod_{|I|=i} \varphi_{K_I}} \Big(\coprod_{|I|=i} C|K_I|\Big).$$

Cone decomposition of $\mathcal{Z}_K(\underline{X})$ by Iriye-Kishimoto

Example

- Let $K = \{1,2\} * \{3,4\}$. Then $\mathbb{R}\mathcal{Z}_K = S^1 \times S^1$



Cone decomposition of $\mathcal{Z}_K(\underline{X})$ by Iriye-Kishimoto

- $\varphi_K \rightsquigarrow \overline{\varphi_K}$ (see [Iriye-Kishimoto 2])
- Theorem[Iriye-Kishimoto 2]

$$\mathcal{Z}_K{}^i(\underline{X}) = \mathcal{Z}_K{}^{i-1}(\underline{X}) \cup_{\overline{\varphi}_{K_I}} (\coprod_{|I|=i} C(|K_I| * \underline{Y}^{*I})).$$

Moreover, $\varphi_K \simeq * \implies \overline{\varphi_K} \simeq *$.

- If $\varphi_{K_I} \simeq *$ for any $I \subset [m]$, then

$$\mathcal{Z}_K(\underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \underline{X}^{\wedge I}.$$

Totally fillable complex relative to $\mathcal{L} = \{L(I)\}$

- Let L be a subcomplex of K,

  and $\mathcal{L} = \{L(I)\}$ be a system of subcomplexes of K.

- Definition (K, L) is fillable if

  there exist a set of minimal non-faces $\{\sigma_1, \dots, \sigma_r\} =: \mathcal{F}(K)$

  such that $(|K \cup \{\sigma_1, \sigma_2, ..., \sigma_r\}|, |L|)$ is a DR-pair.

- Definition K is totally fillable relative to $\mathcal{L}$ if

  $(K_I, L(I))$ is fillable and $\varphi_{L(I)} \simeq *$ for all $I \subset [m]$.

Which K satisfies $\varphi_K \simeq *$?

- Proposition[Iriye-Kishimoto 2] If K is totally fillable, then

$$\varphi_{K_I} \simeq * \text{ for any } I \subset [m].$$

- Proposition If K is totally fillable relative to $\mathcal{L}$, then

$$\varphi_{K_I} \simeq * \text{ for any } I \subset [m].$$

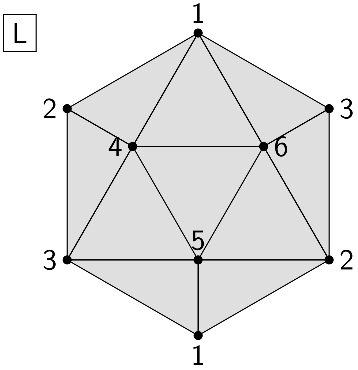
Which K satisfies $\varphi_K \simeq *$?

- $\mathrm{d}(K) := \max\{\operatorname{hodim} K_I \mid, \emptyset \neq I \subset [m]\}$, where $\operatorname{hodim} K_I$ is the homology dimension of $|K_I|$.
- K is n-neighborly if any subset $I \subset [m]$ with $|I| = n+1 \geq 1$ is a simplex of K.
- Proposition[Iriye-Kishimoto 1] If K is $\lceil \mathrm{d}(K)/2 \rceil$-neighborly, then

$$\varphi_{K_I} \simeq * \text{ for any } I \subset [m].$$

Example

- Let $L :=$



Example of decompositon of $\mathcal{Z}_K(\underline{S^1})$

- Let $K := L \sqcup \partial\Delta^{I_1}$ where $I_1 := \{7, 8, 9, 10, 11\}$.
  - K is NOT totally fillable.
  - K is NOT $\lceil \mathrm{d}(K)/2 \rceil$-neighborly.
  - $\exists\ \mathcal{L}$ such that K is totally fillable relative to $\mathcal{L}$.
- From the main theorem, $\mathcal{Z}_K(\underline{S^1})$ is homotopy equivalent to

  $\bigvee_{I_0 \subset I} M(\mathbb{Z}_2, |I|+2) \vee \bigvee_{I_1 \subset I} S^{|I|+4} \vee \bigvee_{I_0 \cap I \in A} S^{|I|+2} \vee \bigvee_{I_0 \cap I_1 \cap I \neq \emptyset} S^{|I|+1}$,

  where

  $A := \{(1,2,3), (1,2,6), (1,3,4), (2,3,5)\} \cup \{J \in \mathcal{P}(I_0)\ ;\ |J| = 4, 5\}$.

Example of decompositon of $\mathcal{Z}_K(\underline{S^1})$

- For $\Sigma^{12}\,\mathbb{R}P^2 = M(\mathbb{Z}_2, 13)$ corresponding to the case of $I = \{1, 2, ...11\}$, the composition

$$\Sigma^{12}\mathbb{R}P^2 \hookrightarrow \mathcal{W}_K(\underline{S^1}) \xrightarrow{\simeq} \mathcal{Z}_K \xrightarrow{w_K} \mathcal{D}J_K$$

is, up to homotopy, $[[\cdots[[w_L, a_7], a_8], \cdots], a_{11}]$, where $a_i : S^2_i \hookrightarrow \mathcal{D}J_K$ is the inclusion.

Reference

- [Iriye and Kishimoto 1] Fat-wedge filtration and decomposition of polyhedral products. Kyoto J. Math. 2019
- [Iriye and Kishimoto 2] Whitehead products in moment-angle complexes. J. Math. Soc. Japan. 2020