Topics about complex and real toric spaces

Zhang, Song

Fudan University

Toric Topology 2022 in Osaka
March 23, 2022

Contents

Introduction

Moment-angle complexes and polyhedral products

Moment-angle complexes are firstly formally formulated by Buchstaber and Panov in 1998:

Definition of a moment-angle complex

$$\mathcal{Z}_K = \mathcal{Z}(K;(D^2,S^1)) = \bigcup_{I\in K}\prod_{i\in I} D_i^2 \times \prod_{i\notin I} S_i^1 \subset (D^2)^m$$

where K is a simplicial complex on vertices set $[m]=\{1,\dots,m\}$.

A generalized functorial construction is a polyhedral product (aka. a generalized moment-angle complex), which has been studied since 1966 in G. Porter's work:

Definition of a polyhedral product

$$\mathcal{Z}(K;(\underline{X},\underline{A})) = (\underline{X},\underline{A})^K = \bigcup_{I\in K}\prod_{i\in I} X_i \times \prod_{i\notin I} A_i$$

where $(\underline{X},\underline{A}) = \{(X_i,A_i) : i=1,\dots,m\}$ is a family of based CW-pairs.

Examples

$$\mathcal{Z}_K = (D^2)^m \text{ when } K=\Delta^{m-1}$$

$$\mathcal{Z}_K = S^{2m-1} \text{ when } K=\partial\Delta^{m-1}$$

Introduction

Moment-angle complexes and polyhedral products

Moment-angle complexes are firstly formally formulated by Buchstaber and Panov in 1998:

Definition of a moment-angle complex

$$\mathcal{Z}_K = \mathcal{Z}(K;(D^2,S^1)) = \bigcup_{I\in K}\prod_{i\in I} D_i^2 \times \prod_{i\notin I} S_i^1 \subset (D^2)^m$$

where K is a simplicial complex on vertices set $[m]=\{1,\ldots,m\}$.

A generalized functorial construction is a polyhedral product (aka. a generalized moment-angle complex), which has been studied since 1966 in G. Porter's work:

Definition of a polyhedral product

$$\mathcal{Z}(K;(\underline{X},\underline{A})) = (\underline{X},\underline{A})^K = \bigcup_{I\in K}\prod_{i\in I} X_i \times \prod_{i\notin I} A_i$$

where $(\underline{X},\underline{A}) = \{(X_i,A_i) : i=1,\ldots,m\}$ is a family of based CW-pairs.

Examples

$$\mathcal{Z}_K = (D^2)^m \text{ when } K = \Delta^{m-1}$$

$$\mathcal{Z}_K = S^{2m-1} \text{ when } K = \partial\Delta^{m-1}$$

Introduction

Moment-angle complexes and polyhedral products

Moment-angle complexes are firstly formally formulated by Buchstaber and Panov in 1998:

Definition of a moment-angle complex

$$\mathcal{Z}_K = \mathcal{Z}(K;(D^2,S^1)) = \bigcup_{I\in K}\prod_{i\in I} D_i^2 \times \prod_{i\notin I} S_i^1 \subset (D^2)^m$$

where K is a simplicial complex on vertices set $[m]=\{1,\ldots,m\}$.

A generalized functorial construction is a polyhedral product (aka. a generalized moment-angle complex), which has been studied since 1966 in G. Porter's work:

Definition of a polyhedral product

$$\mathcal{Z}(K;(\underline{X},\underline{A})) = (\underline{X},\underline{A})^K = \bigcup_{I\in K}\prod_{i\in I} X_i \times \prod_{i\notin I} A_i$$

where $(\underline{X},\underline{A}) = \{(X_i,A_i) : i=1,\ldots,m\}$ is a family of based CW-pairs.

Examples

$$\mathcal{Z}_K = (D^2)^m \text{ when } K=\Delta^{m-1}$$
$$\mathcal{Z}_K = S^{2m-1} \text{ when } K=\partial\Delta^{m-1}$$

Introduction

Cohomology of moment-angle complexes

In what way the cohomology of moment-angle complexes perform?

Definition of Stanley-Reisner algebra (face ring) of K

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \cdots, v_m]/\mathcal{I}_K = \mathbb{Z}[v_1, \cdots, v_m]/\sum_{\{i_1,\cdots,i_s\}\notin K}(v_{i_1}\cdots v_{i_s})$$

The Tor-algebra: by the Koszul resolution of $\mathbb{Z}[K]$, and 1-to-1 correspondence between the cells of $\mathcal{Z}_K$ and the generators of the quotient of the Koszul complex

[Baskakov, Buchstaber, and Panov, 2004]

$$H^*(\mathcal{Z}_K;\mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$$

Furthermore, $H^*(\mathcal{Z}_K)$ can be graded by the Hochster formula

[Buchstaber and Panov, 2002, or in their book *Toric Topology*]

$$H^p(\mathcal{Z}_K) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{p-|I|-1}(K_I)$$

where K_I is the full subcomplex of K on I.

Introduction

Cohomology of moment-angle complexes

In what way the cohomology of moment-angle complexes perform?

Definition of Stanley-Reisner algebra (face ring) of K

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \cdots, v_m]/\mathcal{I}_K = \mathbb{Z}[v_1, \cdots, v_m]/\sum_{\{i_1,\cdots,i_s\}\notin K}(v_{i_1}\cdots v_{i_s})$$

The Tor-algebra: by the Koszul resolution of $\mathbb{Z}[K]$, and 1-to-1 correspondence between the cells of $\mathcal{Z}_K$ and the generators of the quotient of the Koszul complex

[Baskakov, Buchstaber, and Panov, 2004]

$$H^*(\mathcal{Z}_K;\mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$$

Furthermore, $H^*(\mathcal{Z}_K)$ can be graded by the Hochster formula

[Buchstaber and Panov, 2002, or in their book *Toric Topology*]

$$H^p(\mathcal{Z}_K) \cong \bigoplus_{I\subset[m]}\widetilde{H}^{p-|I|-1}(K_I)$$

where K_I is the full subcomplex of K on I.

Introduction

Cohomology of moment-angle complexes

In what way the cohomology of moment-angle complexes perform?

Definition of Stanley-Reisner algebra (face ring) of K

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \cdots, v_m]/\mathcal{I}_K = \mathbb{Z}[v_1, \cdots, v_m]/\sum_{\{i_1, \cdots, i_s\} \notin K}(v_{i_1} \cdots v_{i_s})$$

The Tor-algebra: by the Koszul resolution of $\mathbb{Z}[K]$, and 1-to-1 correspondence between the cells of $\mathcal{Z}_K$ and the generators of the quotient of the Koszul complex

[Baskakov, Buchstaber, and Panov, 2004]

$$H^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1, \cdots, v_m]}(\mathbb{Z}[K], \mathbb{Z})$$

Furthermore, $H^*(\mathcal{Z}_K)$ can be graded by the Hochster formula

[Buchstaber and Panov, 2002, or in their book *Toric Topology*]

$$H^p(\mathcal{Z}_K) \cong \bigoplus_{I \subset [m]} \widetilde{H}^{p-|I|-1}(K_I)$$

where K_I is the full subcomplex of K on I.

Introduction

Cohomology of moment-angle complexes

In what way the cohomology of moment-angle complexes perform?

Definition of Stanley-Reisner algebra (face ring) of K

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \cdots, v_m]/\mathcal{I}_K = \mathbb{Z}[v_1, \cdots, v_m]/\sum_{\{i_1,\cdots,i_s\}\notin K}(v_{i_1}\cdots v_{i_s})$$

The Tor-algebra: by the Koszul resolution of $\mathbb{Z}[K]$, and 1-to-1 correspondence between the cells of $\mathcal{Z}_K$ and the generators of the quotient of the Koszul complex

[Baskakov, Buchstaber, and Panov, 2004]

$$H^*(\mathcal{Z}_K;\mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$$

Furthermore, $H^*(\mathcal{Z}_K)$ can be graded by the Hochster formula

[Buchstaber and Panov, 2002, or in their book *Toric Topology*]

$$H^p(\mathcal{Z}_K) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{p-|I|-1}(K_I)$$

where K_I is the full subcomplex of K on I.

Introduction

Generalizations of $\mathcal{Z}_K = \mathcal{Z}(K; (D^2, S^1))$

Let $\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})} = \mathcal{Z}(K; (\underline{\mathbb{D}}, \underline{\mathbb{S}}))$, where $(\underline{\mathbb{D}}, \underline{\mathbb{S}}) = \{(\mathbb{D}_i, \mathbb{S}_i)\}_{i=1}^m$ satisfy the following:

$$\begin{cases} \widetilde{H}_j(\mathbb{D}_i) = 0 \text{ for any } j, \\ \text{there exists a unique integer } \kappa_i \text{ such that } \widetilde{H}_{\kappa_i}(\mathbb{S}_i) \cong \mathbb{Z} \text{ and } \widetilde{H}_j(\mathbb{S}_i) = 0 \text{ for any } j \neq \kappa_i. \end{cases}$$

[X. Cao and Z. Lü, 2012]

$$H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K], \mathbf{k}),$$

with the Hochster formula

$$H^n(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k}) \cong \bigoplus_{\substack{\mathbf{a} \in \{0,1\}^m \\ -1+\sum_{l\in\mathbf{a}}(\kappa_l+1)=n}} \mathrm{Tor}^{-i,\mathbf{a}}_{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K], \mathbf{k}) \cong \bigoplus_{\substack{\mathbf{a} \in \{0,1\}^m \\ -1+\sum_{l\in\mathbf{a}}(\kappa_l+1)=n}} \widetilde{H}^{|\mathbf{a}|-i-1}(K_{\mathbf{a}}; \mathbf{k})$$

where $\mathbf{k}$ is a field with arbitrary characteristic.

Introduction

Generalizations of $\mathcal{Z}_K = \mathcal{Z}(K; (D^2, S^1))$

Let $\mathcal{Z}_K^{(\underline{\mathbb{D}},\underline{\mathbb{S}})} = \mathcal{Z}(K; (\underline{\mathbb{D}}, \underline{\mathbb{S}}))$, where $(\underline{\mathbb{D}}, \underline{\mathbb{S}}) = \{(\mathbb{D}_i, \mathbb{S}_i)\}_{i=1}^m$ satisfy the following:

$$\begin{cases} \widetilde{H}_j(\mathbb{D}_i) = 0 \text{ for any } j, \\ \text{there exists a unique integer } \kappa_i \text{ such that } \widetilde{H}_{\kappa_i}(\mathbb{S}_i) \cong \mathbb{Z} \text{ and } \widetilde{H}_j(\mathbb{S}_i) = 0 \text{ for any } j \neq \kappa_i. \end{cases}$$

[X. Cao and Z. Lü, 2012]

$$H^*(\mathcal{Z}_K^{(\underline{\mathbb{D}},\underline{\mathbb{S}})}; \mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K], \mathbf{k}),$$

with the Hochster formula

$$H^n(\mathcal{Z}_K^{(\underline{\mathbb{D}},\underline{\mathbb{S}})}; \mathbf{k}) \cong \bigoplus_{\substack{\mathbf{a} \in \{0,1\}^m \\ -1+\sum_{l\in\mathbf{a}}(\kappa_l+1) = n}} \mathrm{Tor}^{-i,\mathbf{a}}_{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K], \mathbf{k}) \cong \bigoplus_{\substack{\mathbf{a} \in \{0,1\}^m \\ -1+\sum_{l\in\mathbf{a}}(\kappa_l+1) = n}} \widetilde{H}^{|\mathbf{a}|-i-1}(K_{\mathbf{a}}; \mathbf{k})$$

where $\mathbf{k}$ is a field with arbitrary characteristic.

Introduction

Generalizations of $\mathcal{Z}_K = \mathcal{Z}(K; (D^2, S^1))$

For a moment-angle complex over the simplicial poset $\mathcal{S}$, given by

$$\mathcal{Z}_{\mathcal{S}} = \bigcup_{\sigma \in \mathcal{S}} \prod_{j \preceq \sigma} D_j^2 \times \prod_{j \not\preceq \sigma} S_j^1 \subset (D^2)^m$$

and the face ring of $\mathcal{S}$ is given by [Stanley, 1991]

$$\mathbb{Z}[\mathcal{S}] = \mathbb{Z}[v_\sigma | \sigma \in \mathcal{S}]/\mathcal{I}_{\mathcal{S}} = \mathbb{Z}[v_\sigma | \sigma \in \mathcal{S}]/\big((v_{\hat{0}} - 1) + \sum_{\sigma,\tau \in \mathcal{S}} (v_\sigma v_\tau - v_{\sigma\wedge\tau} \cdot \sum_{\eta \in \sigma \vee \tau} v_\eta)\big).$$

Still the Tor-algebra and the Hochster formula are available:

[Z. Lü and T. Panov, 2011]

$$H^*(\mathcal{Z}_{\mathcal{S}}; \mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[\mathcal{S}], \mathbb{Z})$$

with the Hochster formula

$$H^p(\mathcal{Z}_{\mathcal{S}}; \mathbb{Z}) \cong \bigoplus_{-i+2|\mathbf{a}|=p} \mathrm{Tor}^{-i,2\mathbf{a}}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[\mathcal{S}], \mathbb{Z}) \cong \bigoplus_{-i+2|\mathbf{a}|=p} \widetilde{H}^{|\mathbf{a}|-i-1}(|\mathcal{S}_{\mathbf{a}}|)$$

Introduction

Generalizations of $\mathcal{Z}_K = \mathcal{Z}(K; (D^2, S^1))$

For a moment-angle complex over the simplicial poset $\mathcal{S}$, given by

$$\mathcal{Z}_{\mathcal{S}} = \bigcup_{\sigma \in \mathcal{S}} \prod_{j \preceq \sigma} D_j^2 \times \prod_{j \npreceq \sigma} S_j^1 \subset (D^2)^m$$

and the face ring of $\mathcal{S}$ is given by [Stanley, 1991]

$$\mathbb{Z}[\mathcal{S}] = \mathbb{Z}[v_\sigma | \sigma \in \mathcal{S}]/\mathcal{I}_{\mathcal{S}} = \mathbb{Z}[v_\sigma | \sigma \in \mathcal{S}]/\big((v_{\hat{0}} - 1) + \sum_{\sigma, \tau \in \mathcal{S}} (v_\sigma v_\tau - v_{\sigma \wedge \tau} \cdot \sum_{\eta \in \sigma \vee \tau} v_\eta)\big).$$

Still the Tor-algebra and the Hochster formula are available:

[Z. Lü and T. Panov, 2011]

$$H^*(\mathcal{Z}_{\mathcal{S}}; \mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1, \cdots, v_m]}(\mathbb{Z}[\mathcal{S}], \mathbb{Z})$$

with the Hochster formula

$$H^p(\mathcal{Z}_{\mathcal{S}}; \mathbb{Z}) \cong \bigoplus_{-i+2|\mathbf{a}|=p} \mathrm{Tor}^{-i,2\mathbf{a}}_{\mathbb{Z}[v_1, \cdots, v_m]}(\mathbb{Z}[\mathcal{S}], \mathbb{Z}) \cong \bigoplus_{-i+2|\mathbf{a}|=p} \widetilde{H}^{|\mathbf{a}|-i-1}(|\mathcal{S}_{\mathbf{a}}|)$$

Introduction

Generalizations of $\mathcal{Z}_K = \mathcal{Z}(K; (D^2, S^1))$

For the general case, there is a stable decomposition of a polyhedral product:

[Bahri, Bendersky, Cohen, and Gitler, 2010]

$$\Sigma(\mathcal{Z}(K; (\underline{X}, \underline{A}))) \simeq \Sigma\Big(\bigvee_{I \subset [m]} \widehat{\mathcal{Z}}(K_I; (\underline{X}, \underline{A})_I)\Big)$$

where Σ means the reduced suspension, $\widehat{\mathcal{Z}}$ means the smash polyhedral product.

which will lead to the decomposition of its cohomology:

[Bahri, Bendersky, Cohen, and Gitler, 2012]

$$H^*(\mathcal{Z}(K; (\underline{X}, \underline{A}))) \cong \bigoplus_{I \subset [m]} H^*(\widehat{\mathcal{Z}}(K_I; (\underline{X}, \underline{A})_I))$$

Introduction

Generalizations of $\mathcal{Z}_K = \mathcal{Z}(K; (D^2, S^1))$

For the general case, there is a stable decomposition of a polyhedral product:

[Bahri, Bendersky, Cohen, and Gitler, 2010]

$$\Sigma(\mathcal{Z}(K; (\underline{X}, \underline{A}))) \simeq \Sigma\Big(\bigvee_{I \subset [m]} \widehat{\mathcal{Z}}(K_I; (\underline{X}, \underline{A})_I) \Big)$$

where Σ means the reduced suspension, $\widehat{\mathcal{Z}}$ means the smash polyhedral product.

which will lead to the decomposition of its cohomology:

[Bahri, Bendersky, Cohen, and Gitler, 2012]

$$H^*(\mathcal{Z}(K; (\underline{X}, \underline{A}))) \cong \bigoplus_{I \subset [m]} H^*(\widehat{\mathcal{Z}}(K_I; (\underline{X}, \underline{A})_I))$$

Introduction

Moment-angle manifolds over simple polytopes

Definition: moment-angle manifolds over P^n with m facets

$$\mathcal{Z}_P = \mathcal{Z}(K_P; (D^2, S^1)) \subset (D^2)^m$$

with a canonical action $(S^1)^m \times \mathcal{Z}_P \to \mathcal{Z}_P$, by

$$(e^{i2\pi\mathbf{t}}, \mathbf{x}) = (e^{i2\pi t_1}, \cdots, e^{i2\pi t_m}, x_1, \cdots, x_m) \mapsto (e^{i2\pi t_1} \cdot x_1, \cdots, e^{i2\pi t_m} \cdot x_m)$$

where K_P is the boundary complex of the dual of P^n.

An equivalent way to construct $\mathcal{Z}_P$ by the trivial colouring on P^n [*Toric Topology*, section 6.2]

$$(S^1)^m \times P^n / \sim \overset{(S^1)^m\text{-equivariantly}}{\approx} \mathcal{Z}_P$$

constructed by a characteristic function $\lambda : \mathcal{F}(P) \to \mathbb{Z}^m$, $F_i \mapsto e_i$, and

$$(e^{i2\pi\mathbf{t}}, p) \sim (e^{i2\pi\mathbf{t}'}, p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ \forall i \notin \{i_1, \ldots, i_l\}, \quad t'_i - t_i \equiv 0 \ (\mathrm{mod}\ \mathbb{Z}) \end{cases}$$

An example: when $P^1 = [0, 1]$, K_P consists of only 2 vertices

$\mathcal{Z}_P = D^2 \times S^1 \cup S^1 \times D^2 \approx S^3$, and $(S^1)^2 \times P^1 / \sim$ is constructed by colouring $(1, 0)$ on 0 and $(0, 1)$ on 1.

Introduction

Moment-angle manifolds over simple polytopes

Definition: moment-angle manifolds over P^n with m facets

$$\mathcal{Z}_P = \mathcal{Z}(K_P; (D^2, S^1)) \subset (D^2)^m$$

with a canonical action $(S^1)^m \times \mathcal{Z}_P \to \mathcal{Z}_P$, by

$$(e^{i2\pi\mathbf{t}}, \mathbf{x}) = (e^{i2\pi t_1}, \cdots, e^{i2\pi t_m}, x_1, \cdots, x_m) \mapsto (e^{i2\pi t_1} \cdot x_1, \cdots, e^{i2\pi t_m} \cdot x_m)$$

where K_P is the boundary complex of the dual of P^n.

An equivalent way to construct $\mathcal{Z}_P$ by the trivial colouring on P^n [*Toric Topology*, section 6.2]

$$(S^1)^m \times P^n / \sim \overset{(S^1)^m\text{-equivariantly}}{\approx} \mathcal{Z}_P$$

constructed by a characteristic function $\lambda : \mathcal{F}(P) \to \mathbb{Z}^m$, $F_i \mapsto e_i$, and

$$(e^{i2\pi\mathbf{t}}, p) \sim (e^{i2\pi\mathbf{t}'}, p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ \forall i \notin \{i_1, \ldots, i_l\}, \quad t_i' - t_i \equiv 0 \ (\mathrm{mod}\ \mathbb{Z}) \end{cases}$$

An example: when $P^1 = [0,1]$, K_P consists of only 2 vertices

$\mathcal{Z}_P = D^2 \times S^1 \cup S^1 \times D^2 \approx S^3$, and $(S^1)^2 \times P^1 / \sim$ is constructed by colouring $(1,0)$ on 0 and $(0,1)$ on 1.

Introduction

Moment-angle manifolds over simple polytopes

Definition: moment-angle manifolds over P^n with m facets

$$\mathcal{Z}_P = \mathcal{Z}(K_P; (D^2, S^1)) \subset (D^2)^m$$

with a canonical action $(S^1)^m \times \mathcal{Z}_P \to \mathcal{Z}_P$, by

$$(e^{i2\pi \mathbf{t}}, \mathbf{x}) = (e^{i2\pi t_1}, \cdots, e^{i2\pi t_m}, x_1, \cdots, x_m) \mapsto (e^{i2\pi t_1} \cdot x_1, \cdots, e^{i2\pi t_m} \cdot x_m)$$

where K_P is the boundary complex of the dual of P^n.

An equivalent way to construct $\mathcal{Z}_P$ by the trivial colouring on P^n [*Toric Topology*, section 6.2]

$$(S^1)^m \times P^n / \sim \overset{(S^1)^m\text{-equivariantly}}{\approx} \mathcal{Z}_P$$

constructed by a characteristic function $\lambda : \mathcal{F}(P) \to \mathbb{Z}^m$, $F_i \mapsto e_i$, and

$$(e^{i2\pi \mathbf{t}}, p) \sim (e^{i2\pi \mathbf{t}'}, p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ \forall i \notin \{i_1, \dots, i_l\}, \quad t_i' - t_i \equiv 0 \ (\mathrm{mod}\ \mathbb{Z}) \end{cases}$$

An example: when $P^1 = [0, 1]$, K_P consists of only 2 vertices

$\mathcal{Z}_P = D^2 \times S^1 \cup S^1 \times D^2 \approx S^3$, and $(S^1)^2 \times P^1 / \sim$ is constructed by colouring $(1, 0)$ on 0 and $(0, 1)$ on 1.

Introduction

Moment-angle manifolds over simple polytopes

What about the real case?

Definition: the real version

A real moment-angle complex: $\mathbb{R}\mathcal{Z}_K = \mathcal{Z}(K;(D^1,S^0)) = \bigcup_{I\in K} \prod_{i\in I} D^1_i \times \prod_{i\notin I} S^0_i \subset (D^1)^m$, and a real moment-angle manifold over P^n: $\mathbb{R}\mathcal{Z}_P = \mathcal{Z}(K_P;(D^1,S^0)) \subset (D^1)^m$, with the canonical group action $(\mathbb{Z}/2)^m \times \mathbb{R}\mathcal{Z}_P \to \mathbb{R}\mathcal{Z}_P$ given by

$$((\cdots,0_i,\cdots,1_j,\cdots),(\cdots,x_i,\cdots,x_j,\cdots)) \mapsto (\cdots,x_i,\cdots,-x_j,\cdots)$$

An equivalent way to construct $\mathbb{R}\mathcal{Z}_P$ [*Toric Topology*, section 6.2]

$$(\mathbb{Z}/2)^m \times P^n/\sim \overset{(\mathbb{Z}/2)^m\text{-equivariantly}}{\approx} \mathbb{R}\mathcal{Z}_P$$

$$\text{where } (g,p)\sim(g',p') \Leftrightarrow \begin{cases} p=p' \in \mathrm{Int}(F_{i_1}\cap\cdots\cap F_{i_l}) \\ g'-g \in \mathrm{Span}\{e_{i_1},\ldots,e_{i_l}\} \end{cases}$$

An example: when $P^1=[0,1]$, K_P consists of only 2 vertices

$\mathbb{R}\mathcal{Z}_P = D^1\times S^0 \cup S^0\times D^1 \approx S^1$, and $(\mathbb{Z}/2)^2\times P^1/\sim$ is constructed by colouring $(1,0)$ on 0 and $(0,1)$ on 1.

Introduction

Moment-angle manifolds over simple polytopes

What about the real case?

Definition: the real version

A real moment-angle complex: $\mathbb{R}\mathcal{Z}_K = \mathcal{Z}(K; (D^1, S^0)) = \bigcup_{I \in K} \prod_{i \in I} D_i^1 \times \prod_{i \notin I} S_i^0 \subset (D^1)^m$, and a real moment-angle manifold over P^n: $\mathbb{R}\mathcal{Z}_P = \mathcal{Z}(K_P; (D^1, S^0)) \subset (D^1)^m$, with the canonical group action $(\mathbb{Z}/2)^m \times \mathbb{R}\mathcal{Z}_P \to \mathbb{R}\mathcal{Z}_P$ given by

$$((\cdots, 0_i, \cdots, 1_j, \cdots), (\cdots, x_i, \cdots, x_j, \cdots)) \mapsto (\cdots, x_i, \cdots, -x_j, \cdots)$$

An equivalent way to construct $\mathbb{R}\mathcal{Z}_P$ [*Toric Topology*, section 6.2]

$$(\mathbb{Z}/2)^m \times P^n / \sim \overset{(\mathbb{Z}/2)^m\text{-equivariantly}}{\approx} \mathbb{R}\mathcal{Z}_P$$

where $(g, p) \sim (g', p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ g' - g \in \mathrm{Span}\{e_{i_1}, \ldots, e_{i_l}\} \end{cases}$

An example: when $P^1 = [0, 1]$, K_P consists of only 2 vertices

$\mathbb{R}\mathcal{Z}_P = D^1 \times S^0 \cup S^0 \times D^1 \approx S^1$, and $(\mathbb{Z}/2)^2 \times P^1 / \sim$ is constructed by colouring $(1, 0)$ on 0 and $(0, 1)$ on 1.

Introduction

Moment-angle manifolds over simple polytopes

What about the real case?

Definition: the real version

A real moment-angle complex: $\mathbb{R}\mathcal{Z}_K = \mathcal{Z}(K;(D^1,S^0)) = \bigcup_{I\in K} \prod_{i\in I} D^1_i \times \prod_{i\notin I} S^0_i \subset (D^1)^m$, and a real moment-angle manifold over P^n: $\mathbb{R}\mathcal{Z}_P = \mathcal{Z}(K_P;(D^1,S^0)) \subset (D^1)^m$, with the canonical group action $(\mathbb{Z}/2)^m \times \mathbb{R}\mathcal{Z}_P \to \mathbb{R}\mathcal{Z}_P$ given by

$$((\cdots,0_i,\cdots,1_j,\cdots),(\cdots,x_i,\cdots,x_j,\cdots)) \mapsto (\cdots,x_i,\cdots,-x_j,\cdots)$$

An equivalent way to construct $\mathbb{R}\mathcal{Z}_P$ [*Toric Topology*, section 6.2]

$$(\mathbb{Z}/2)^m \times P^n/\sim \overset{(\mathbb{Z}/2)^m\text{-equivariantly}}{\approx} \mathbb{R}\mathcal{Z}_P$$

where $(g,p)\sim(g',p') \Leftrightarrow \begin{cases} p=p' \in \mathrm{Int}(F_{i_1}\cap\cdots\cap F_{i_l}) \\ g'-g \in \mathrm{Span}\{e_{i_1},\ldots,e_{i_l}\} \end{cases}$

An example: when $P^1=[0,1]$, K_P consists of only 2 vertices

$\mathbb{R}\mathcal{Z}_P = D^1\times S^0 \cup S^0\times D^1 \approx S^1$, and $(\mathbb{Z}/2)^2\times P^1/\sim$ is constructed by colouring $(1,0)$ on 0 and $(0,1)$ on 1.

Introduction

Quotients of moment-angle manifolds

Definition: Quotients of moment-angle manifolds, introduced by Panov in 2015

A k-subtorus T^H freely acts on $\mathcal{Z}_P$ will associate with a quotient moment-angle $(m+n-k)$-manifold $\mathcal{Z}_P/T^H$, where $H = (\mathbf{h}_1, \cdots, \mathbf{h}_k)$ is an $m \times k$ integral matrix, and

$$T^H = \{e^{i2\pi(\phi_1\mathbf{h}_1+\cdots+\phi_k\mathbf{h}_k)} : \phi_1, \ldots, \phi_k \in \mathbb{R}\} \subset (S^1)^m.$$

The property which H must satisfy is given in *Toric Topology* (Lemma 4.8.4):

T^H acts on $\mathcal{Z}_P$ freely, if and only if for any simplex $\{i_1, \ldots, i_l\} \in K_P$, vectors $e_{i_1}, \ldots, e_{i_l}, \mathbf{h}_1, \ldots, \mathbf{h}_k$ are unimodular in $\mathbb{Z}^m$.

By the Eilenberg–Moore spectral sequence of the homotopy fibration $\mathcal{Z}_K/T^H \to BT^K \to B(T^m/T^H)$,

[Panov, 2015]

$$H^*(\mathcal{Z}_K/T^H; R) \cong \mathrm{Tor}_{H^*(B(T^m/T^H);R)}(R(K), R)$$

Introduction

Quotients of moment-angle manifolds

Definition: Quotients of moment-angle manifolds, introduced by Panov in 2015

A k-subtorus T^H freely acts on $\mathcal{Z}_P$ will associate with a quotient moment-angle $(m+n-k)$-manifold $\mathcal{Z}_P/T^H$, where $H = (\mathbf{h}_1, \cdots, \mathbf{h}_k)$ is an $m \times k$ integral matrix, and

$$T^H = \{e^{i2\pi(\phi_1\mathbf{h}_1+\cdots+\phi_k\mathbf{h}_k)} : \phi_1, \ldots, \phi_k \in \mathbb{R}\} \subset (S^1)^m.$$

The property which H must satisfy is given in *Toric Topology* (Lemma 4.8.4):

T^H acts on $\mathcal{Z}_P$ freely, if and only if for any simplex $\{i_1, \ldots, i_l\} \in K_P$, vectors $e_{i_1}, \ldots, e_{i_l}, \mathbf{h}_1, \ldots, \mathbf{h}_k$ are unimodular in $\mathbb{Z}^m$.

By the Eilenberg–Moore spectral sequence of the homotopy fibration $\mathcal{Z}_K/T^H \to BT^K \to B(T^m/T^H)$,

[Panov, 2015]

$$H^*(\mathcal{Z}_K/T^H; R) \cong \mathrm{Tor}_{H^*(B(T^m/T^H);R)}(R(K), R)$$

Introduction

Quotients of moment-angle manifolds

Any other simpler way of construction?

A non-degenerate colouring on P^n

A homomorphism $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ satisfies that for every vertex $F_{i_1} \cap \cdots \cap F_{i_n}$, Span$\{\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_n})\}$ is an n-subspace which is a direct summand of $\mathbb{Z}^m$.

Each Λ will determine an $(n+m-k)$-manifold $M(P^n, \Lambda) = (S^1)^{m-k} \times P^n / \sim_\Lambda$, where

$$(e^{i2\pi\mathbf{t}}, p) \sim (e^{i2\pi\mathbf{t}'}, p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ e^{i2\pi\mathbf{t}'} \cdot e^{-i2\pi\mathbf{t}} \in T^{(\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_l}))} \end{cases}$$

An equivalent way of construction

Each $m \times k$ integral matrix H such that T^H acts freely on $\mathcal{Z}_P$ uniquely determines a non-degenerate colouring $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ which is an epimorphism, and vice versa. Furthermore, under this situation,

$$M(P^n, \Lambda) \overset{(S^1)^{m-k} \cong (S^1)^m / T^H\text{-equivariantly}}{\approx} \mathcal{Z}_P / T^H$$

Introduction

Quotients of moment-angle manifolds

Any other simpler way of construction?

A non-degenerate colouring on P^n

A homomorphism $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ satisfies that for every vertex $F_{i_1} \cap \cdots \cap F_{i_n}$, Span$\{\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_n})\}$ is an n-subspace which is a direct summand of $\mathbb{Z}^m$.

Each Λ will determine an $(n+m-k)$-manifold $M(P^n, \Lambda) = (S^1)^{m-k} \times P^n / \sim_\Lambda$, where

$$(e^{i2\pi\mathbf{t}}, p) \sim (e^{i2\pi\mathbf{t}'}, p') \Leftrightarrow \begin{cases} p = p' \in \text{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ e^{i2\pi\mathbf{t}'} \cdot e^{-i2\pi\mathbf{t}} \in T^{(\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_l}))} \end{cases}$$

An equivalent way of construction

Each $m \times k$ integral matrix H such that T^H acts freely on $\mathcal{Z}_P$ uniquely determines a non-degenerate colouring $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ which is an epimorphism, and vice versa. Furthermore, under this situation,

$$M(P^n, \Lambda) \overset{(S^1)^{m-k} \cong (S^1)^m / T^H\text{-equivariantly}}{\approx} \mathcal{Z}_P / T^H$$

Introduction

Quotients of moment-angle manifolds

Any other simpler way of construction?

A non-degenerate colouring on P^n

A homomorphism $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ satisfies that for every vertex $F_{i_1} \cap \cdots \cap F_{i_n}$, Span$\{\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_n})\}$ is an n-subspace which is a direct summand of $\mathbb{Z}^m$.

Each Λ will determine an $(n+m-k)$-manifold $M(P^n, \Lambda) = (S^1)^{m-k} \times P^n / \sim_\Lambda$, where

$$(e^{i2\pi\mathbf{t}}, p) \sim (e^{i2\pi\mathbf{t}'}, p') \Leftrightarrow \begin{cases} p = p' \in \text{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ e^{i2\pi\mathbf{t}'} \cdot e^{-i2\pi\mathbf{t}} \in T^{(\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_l}))} \end{cases}$$

An equivalent way of construction

Each $m \times k$ integral matrix H such that T^H acts freely on $\mathcal{Z}_P$ uniquely determines a non-degenerate colouring $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ which is an epimorphism, and vice versa. Furthermore, under this situation,

$$M(P^n, \Lambda) \overset{(S^1)^{m-k} \cong (S^1)^m / T^H\text{-equivariantly}}{\approx} \mathcal{Z}_P / T^H$$

Introduction

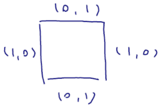
Quasi-toric manifolds

Quasi-toric manifolds $M(P^n, \Lambda)$: introduced by Davis and Januszkiewicz in 1991, a special class of quotients of moment-angle manifolds, constucted by $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^n$.

[Davis and Januszkiewicz, 1991]

$$H^*(M(P^n, \Lambda); \mathbb{Z}) \cong \mathbb{Z}[v_1, \cdots, v_m] \;/\; I + J$$

where $J = \sum_{i=1}^n (\Lambda_{i1} v_1 + \cdots + \Lambda_{im} v_m)$.

$P^2 =$ 

$\Rightarrow M(P^2, \Lambda) \cong S^2 \times S^2$

$$\mathbb{Z}[v_1, v_2, v_3, v_4] \;/\; \underbrace{(v_1 v_3) + (v_2 v_4)}_{I} + \underbrace{(v_1 + v_3) + (v_2 + v_4)}_{J}$$

$$\cong \mathbb{Z}[v_1, v_2] \;/\; (v_1^2) + (v_2^2) \cong H^*(S^2 \times S^2)$$

Introduction

Real toric spaces over simple polytopes

What about the real case?

Definition: a real toric space over P^n

Each $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^{m-k}$ which satisfies the condition

$$\text{for every vertex } F_{i_1} \cap \cdots \cap F_{i_n},\ \lambda(F_{i_1}), \ldots, \lambda(F_{i_n}) \text{ are linearly independent in } (\mathbb{Z}/2)^{m-k}$$

will associate with a real toric space $M_\lambda = (\mathbb{Z}/2)^{m-k} \times P^n / \sim_\lambda$, where

$$(g,p) \sim (g',p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ g' - g \in \mathrm{Span}\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_l})\} \end{cases}$$

Similarly to the complex case,

A rank-k-subgroup $H \subset (\mathbb{Z}/2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_P$ will give rise to an n-manifold $\mathbb{R}\mathcal{Z}_P/H$.

Introduction

Real toric spaces over simple polytopes

What about the real case?

Definition: a real toric space over P^n

Each $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^{m-k}$ which satisfies the condition

$$\text{for every vertex } F_{i_1} \cap \cdots \cap F_{i_n},\ \lambda(F_{i_1}), \ldots, \lambda(F_{i_n}) \text{ are linearly independent in } (\mathbb{Z}/2)^{m-k}$$

will associate with a real toric space $M_\lambda = (\mathbb{Z}/2)^{m-k} \times P^n / \sim_\lambda$, where

$$(g, p) \sim (g', p') \Leftrightarrow \begin{cases} p = p' \in \mathrm{Int}(F_{i_1} \cap \cdots \cap F_{i_l}) \\ g' - g \in \mathrm{Span}\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_l})\} \end{cases}$$

Similarly to the complex case,

A rank-k-subgroup $H \subset (\mathbb{Z}/2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_P$ will give rise to an n-manifold $\mathbb{R}\mathcal{Z}_P/H$.

Introduction

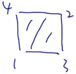
Real toric spaces over simple polytopes

An equivalent way of construction

Each k-subspace H of $(\mathbb{Z}/2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_P$ uniquely determines a non-degenerate colouring $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^{m-k}$ which is an epimorphism, and vice versa. Furthermore, under this situation,

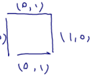
$$M_\lambda \overset{(\mathbb{Z}/2)^{m-k}\cong(\mathbb{Z}/2)^m/H\text{-equivariantly}}{\approx} \mathbb{R}\mathcal{Z}_P/H$$

Small covers: Introduced by Davis and Januszkiewicz in 1991, a special class of real toric spaces, for they can be constructed by non-degenerate $(\mathbb{Z}/2)^n$-colourings, or equivalently, orbit spaces of $\mathbb{R}\mathcal{Z}_P$ by free $(\mathbb{Z}/2)^{m-n}$-actions.

$P^2 =$ 

$\Rightarrow \mathbb{R}\mathcal{Z}_P = D^1\times S^0\times D^1\times S^0 \cup S^0\times D^1\times D^1\times S^0 \cup (S^0\times D^1\times S^0\times D^1) \cup (D^1\times S^0\times S^0\times D^1)$

Let $H = \mathrm{Span}\{e_1-e_2,\ e_3-e_4\} \Rightarrow \mathbb{R}\mathcal{Z}_P/H \approx S^1\times S^1$

Let $\lambda =$ 

$\Rightarrow M_\lambda \approx T^2$

Introduction

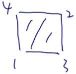
Real toric spaces over simple polytopes

An equivalent way of construction

Each k-subspace H of $(\mathbb{Z}/2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_P$ uniquely determines a non-degenerate colouring $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^{m-k}$ which is an epimorphism, and vice versa. Furthermore, under this situation,

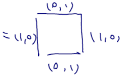
$$M_\lambda \quad \overset{(\mathbb{Z}/2)^{m-k}\cong(\mathbb{Z}/2)^m/H\text{-equivariantly}}{\approx} \quad \mathbb{R}\mathcal{Z}_P/H$$

Small covers: Introduced by Davis and Januszkiewicz in 1991, a special class of real toric spaces, for they can be constructed by non-degenerate $(\mathbb{Z}/2)^n$-colourings, or equivalently, orbit spaces of $\mathbb{R}\mathcal{Z}_P$ by free $(\mathbb{Z}/2)^{m-n}$-actions.

$P^2 =$ 

$\Rightarrow \mathbb{R}\mathcal{Z}_P = D^1\times S^0\times D^1\times S^0 \cup S^0\times D^1\times D^1\times S^0$
$\cup (S^0\times D^1\times S^0\times D^1) \cup (D^1\times S^0\times S^0\times D^1)$

Let $H = \mathrm{Span}\{e_1-e_2, e_3-e_4\} \Rightarrow \mathbb{R}\mathcal{Z}_P/H \approx S^1\times S^1$

Let $\lambda =$ 

$\Rightarrow M_\lambda \approx T^2$

Filtrations of complex and real toric spaces

The cubical decomposition of P^n

[Buchstaber and Panov, 2002 or *Toric Topology*]

$$\text{Define } c_P : P^n \to \partial\mathbb{I}^m = \partial[0,1]^m$$

by assigning

$$\begin{cases} \text{the barycentre of each proper face } f = F_{i_1} \cap \cdots \cap F_{i_l} \mapsto (\epsilon_1, \ldots, \epsilon_m) \\ \text{where } \epsilon_i = 0 \text{ if } i \in \{i_1, \ldots, i_l\}, \text{ otherwise } \epsilon_i = 1, \\ \text{the barycentre of } P^n \mapsto (1, \ldots, 1) \end{cases}$$

and extending this assignment linearly on all the simplices of barycentric subdivision of P^n.

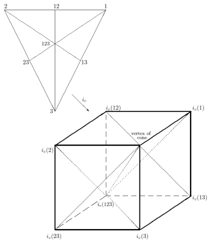


FIGURE 2.11. Taking cone over the barycentric subdivision of simplex defines a triangulation of the cube.

Filtrations of complex and real toric spaces

The cubical decomposition of P^n

Notation

Let $\square_f = c_P(f) = \{1\} \times \cdots \times \underset{i_1\text{-th}}{[0,1]} \times \cdots \times \underset{i_l\text{-th}}{[0,1]} \times \cdots \times \{1\}$,

$\widehat{\square}_f = \{1\} \times \cdots \times \underset{i_1\text{-th}}{[0,1)} \times \cdots \times \underset{i_l\text{-th}}{[0,1)} \times \cdots \times \{1\}$, and $\square_P = \{1\} \times \cdots \times \{1\}$. And topologically we consider the quotient of moment-angle manifold $M(P^n, \Lambda)$ and the real toric space M_λ as $(S^1)^r \times c_P(P^n)/\sim_\Lambda$ and $(\mathbb{Z}/2)^r \times c_P(P^n)/\sim_\lambda$ respectively, with the given equivalent relation

$$(e^{i2\pi \mathbf{t}}, \mathbf{y}) \sim (e^{i2\pi \mathbf{t}'}, \mathbf{y}') \Leftrightarrow \begin{cases} \mathbf{y} = \mathbf{y}' \text{ and } y_{i_1} = \cdots = y_{i_l} = 0,\ y_i > 0 \text{ if } i \notin \{i_1, \ldots, i_l\} \\ e^{i2\pi \mathbf{t}'} \cdot e^{-i2\pi \mathbf{t}} \in T^{(\Lambda(e_{i_1}), \cdots, \Lambda(e_{i_l}))} \end{cases}$$

and

$$(g, \mathbf{y}) \sim (g', \mathbf{y}') \Leftrightarrow \begin{cases} \mathbf{y} = \mathbf{y}' \text{ and } y_{i_1} = \cdots = y_{i_l} = 0,\ y_i > 0 \text{ if } i \notin \{i_1, \ldots, i_l\} \\ g' - g \in \text{Span}\{\lambda(e_{i_1}), \ldots, \lambda(e_{i_l})\} \end{cases}$$

respectively.

Filtrations of complex and real toric spaces

A filtration of a quotient of moment-angle manifold $M(P^n, \Lambda)$

Given $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^r$, for each proper face $f = F_{i_1} \cap \cdots \cap F_{i_l}$ of P^n, choose column vectors $\alpha_{l+1}, \dots, \alpha_r \in \mathbb{Z}^r$ such that $\Lambda_f = \left(\lambda_{i_1}, \cdots, \lambda_{i_l}, \alpha_{l+1}, \cdots, \alpha_r\right)$ has the determinant ± 1. Let
$\Phi_f : (D^2)^l \times (S^1)^{r-l} \to \pi^{-1}(\square_f) = (S^1)^r \times \square_f / \sim_\Lambda$, by

$$\left(y_{i_1} e^{i2\pi t_1}, \cdots, y_{i_l} e^{i2\pi t_l}, e^{i2\pi t_{l+1}}, \cdots, e^{i2\pi t_r}\right) \mapsto \left[e^{i2\pi \Lambda_f \cdot \mathbf{t}}, 1, \cdots, y_{i_1}, \cdots, y_{i_l}, \cdots, 1\right].$$

It is easy to verify that

$$\Phi_f \text{ is surjective, besides } \Phi_f \big|_{(\mathrm{Int} D^2)^l \times (S^1)^{r-l}} : (\mathrm{Int} D^2)^l \times (S^1)^{r-l} \approx \pi^{-1}(\widehat{\square}_f) = (S^1)^r \times \widehat{\square}_f / \sim_\Lambda,$$

for any face g of f such that $c_P(\hat{g}) = (1, \cdots, \underset{i_1\text{-th}}{0}, \cdots, \underset{i_{l_0}\text{-th}}{1}, \cdots, \underset{i_l\text{-th}}{0}, \cdots, 1)$,

$$\Phi_f\left((D^2)^{l_0 - 1} \times S^1 \times (D^2)^{l - l_0} \times (S^1)^{r-l}\right) = \Phi_g\left((D^2)^{l-1} \times (S^1)^{r-l+1}\right).$$

Filtrations of complex and real toric spaces

A filtration of a quotient of moment-angle manifold $M(P^n, \Lambda)$

Given $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^r$, for each proper face $f = F_{i_1} \cap \cdots \cap F_{i_l}$ of P^n, choose column vectors $\alpha_{l+1}, \ldots, \alpha_r \in \mathbb{Z}^r$ such that $\Lambda_f = (\lambda_{i_1}, \cdots, \lambda_{i_l}, \alpha_{l+1}, \cdots, \alpha_r)$ has the determinant ± 1. Let $\Phi_f : (D^2)^l \times (S^1)^{r-l} \to \pi^{-1}(\Box_f) = (S^1)^r \times \Box_f / \sim_\Lambda$, by

$$\left(y_{i_1} e^{i2\pi t_1}, \cdots, y_{i_l} e^{i2\pi t_l}, e^{i2\pi t_{l+1}}, \cdots, e^{i2\pi t_r}\right) \mapsto \left[e^{i2\pi \Lambda_f \cdot \mathbf{t}}, 1, \cdots, y_{i_1}, \cdots, y_{i_l}, \cdots, 1\right].$$

It is easy to verify that

$$\Phi_f \text{ is surjective, besides } \Phi_f \big|_{(\mathrm{Int}D^2)^l \times (S^1)^{r-l}} : (\mathrm{Int}D^2)^l \times (S^1)^{r-l} \approx \pi^{-1}(\widehat{\Box}_f) = (S^1)^r \times \widehat{\Box}_f / \sim_\Lambda,$$

for any face g of f such that $c_P(\hat{g}) = (1, \cdots, \underset{i_1\text{-th}}{0}, \cdots, \underset{i_{l_0}\text{-th}}{1}, \cdots, \underset{i_l\text{-th}}{0}, \cdots, 1)$,

$$\Phi_f\left((D^2)^{l_0-1} \times S^1 \times (D^2)^{l-l_0} \times (S^1)^{r-l}\right) = \Phi_g\left((D^2)^{l-1} \times (S^1)^{r-l+1}\right).$$

Filtrations of complex and real toric spaces

A filtration of a quotient of moment-angle manifold $M(P^n, \Lambda)$

Thus we are able to endow a filtration to the manifold $M(P^n, \lambda)$:

$$X^0 \hookrightarrow \cdots \hookrightarrow X^r \hookrightarrow \cdots \hookrightarrow X^{r+n} = M(P^n, \lambda)$$

where, for $0 \leq l \leq n$, $X^{r+l} = \bigcup_{\dim f = n-l} \Phi_f((D^2)^l \times (S^1)^{r-l}) = \bigcup_{\dim f = n-l} \pi^{-1}(\square_f)$; for $0 \leq j \leq r$, X^j is the j-th cellular skeleton of $X^r \approx (S^1)^r$, which is decomposed canonically, i.e., take one 1-cell and one 0-cell $\{1\}$ for S^1. Notice that formally, $X^0 = \{1\} \times \cdots \times \{1\} \times (1, \cdots, 1)/\sim_\Lambda$ and $X^r = (S^1)^r \times (1, \cdots, 1)/\sim_\Lambda$.

Filtrations of complex and real toric spaces

A cellular decomposition of a real toric space M_λ

Definition: the f-vector of P^n [*An Introduction to Convex Polytopes*, Bronsted]

$$f(P) = (f_0, \cdots, f_{n-1})$$

where f_i is the number of i-faces of P^n. That is, f_0 is the number of vertices of P^n, and $f_{n-1} = m$. In addition we let $f_n = 1$.

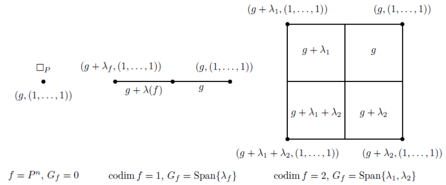
Then given $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^r$, we have

The cellular structure of M_λ

The n-dimensional CW-complex M_λ has l-cells with the number of $f_{n-l} \cdot 2^{r-l}$, for $l = 0, \ldots, n$.

Let $\Theta : (\mathbb{Z}/2)^r \times P^n \to M_\lambda$, $G_f =$Span$\{\lambda(e_{i_1}), \ldots, \lambda(e_{i_l})\}$, and $C_f^{(g)} = \bigcup_{h \in g+G_f} \Theta(h \times \square_f)$. Then

$C_f^{(g)} \approx [-1, 1]^{n-\dim f}$, for example:



Filtrations of complex and real toric spaces

A cellular decomposition of a real toric space M_λ

Definition: the f-vector of P^n [*An Introduction to Convex Polytopes*, Bronsted]

$$f(P) = (f_0, \cdots, f_{n-1})$$

where f_i is the number of i-faces of P^n. That is, f_0 is the number of vertices of P^n, and $f_{n-1} = m$. In addition we let $f_n = 1$.

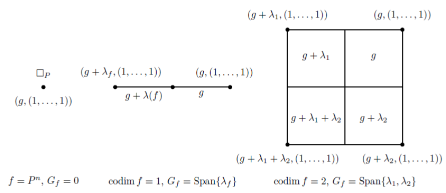
Then given $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^r$, we have

The cellular structure of M_λ

The n-dimensional CW-complex M_λ has l-cells with the number of $f_{n-l} \cdot 2^{r-l}$, for $l = 0, \ldots, n$.

Let $\Theta : (\mathbb{Z}/2)^r \times P^n \to M_\lambda$, $G_f =$Span$\{\lambda(e_{i_1}), \ldots, \lambda(e_{i_l})\}$, and $C_f^{(g)} = \bigcup_{h \in g+G_f} \Theta(h \times \Box_f)$. Then

$C_f^{(g)} \approx [-1,1]^{n-\dim f}$, for example:



Filtrations of complex and real toric spaces

A cellular decomposition of a real toric space M_λ

Definition: the f-vector of P^n [*An Introduction to Convex Polytopes*, Bronsted]

$$f(P) = (f_0, \cdots, f_{n-1})$$

where f_i is the number of i-faces of P^n. That is, f_0 is the number of vertices of P^n, and $f_{n-1} = m$. In addition we let $f_n = 1$.

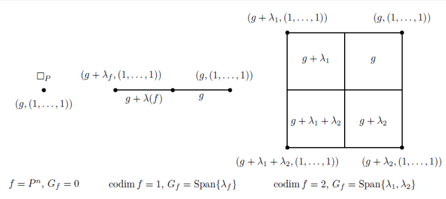
Then given $\lambda : (\mathbb{Z}/2)^m \to (\mathbb{Z}/2)^r$, we have

The cellular structure of M_λ

The n-dimensional CW-complex M_λ has l-cells with the number of $f_{n-l} \cdot 2^{r-l}$, for $l = 0, \ldots, n$.

Let $\Theta : (\mathbb{Z}/2)^r \times P^n \to M_\lambda$, $G_f =$Span$\{\lambda(e_{i_1}), \ldots, \lambda(e_{i_l})\}$, and $C_f^{(g)} = \bigcup_{h \in g+G_f} \Theta(h \times \square_f)$. Then

$C_f^{(g)} \approx [-1,1]^{n-\dim f}$, for example:



The cohomology of quotients of moment-angle manifolds over polygons

The induced spectral sequence by the filtration

Review

Choose $\alpha_{l+1}, \dots, \alpha_r \in \mathbb{Z}^r$ such that $|\Lambda_f = (\lambda_{i_1}, \cdots, \lambda_{i_l}, \alpha_{l+1}, \cdots, \alpha_r)| = 1$, and

$$\Phi_f : (D^2)^l \times (S^1)^{r-l} \to \pi^{-1}(\square_f) = (S^1)^r \times \square_f / \sim_\Lambda$$

and

$$X^0 \hookrightarrow \cdots \hookrightarrow X^r \hookrightarrow \cdots \hookrightarrow X^{r+n} = M(P^n, \lambda).$$

The (co)homology of the pair (X^{r+l}, X^{r+l-1})

There is an isomorphism

$$\bigoplus_{\dim f = n-l} \Phi_{f*} : \bigoplus_{\dim f = n-l} H_p((D^2)^l \times (S^1)^{r-l}, \partial(D^2)^l \times (S^1)^{r-l}) \xrightarrow{\cong} H_p(X^{r+l}, X^{r+l-1}), \quad \forall p \geq 0$$

Thus we have the description

$$H_j(X^{r+l}, X^{r+l-1}) \cong H^j(X^{r+l}, X^{r+l-1}; \mathbb{Z}) \cong \begin{cases} \bigoplus_{\dim f = n-l} \mathbb{Z}^{\binom{r-l}{j-2l}}, & \text{when } 2l \leq j \leq r+l \\ 0, & \text{when } j < 2l \text{ or } j > r+l \end{cases}$$

The cohomology of quotients of moment-angle manifolds over polygons

The induced spectral sequence by the filtration

Review

Choose $\alpha_{l+1}, \ldots, \alpha_r \in \mathbb{Z}^r$ such that $|\Lambda_f = (\lambda_{i_1}, \cdots, \lambda_{i_l}, \alpha_{l+1}, \cdots, \alpha_r)| = 1$, and

$$\Phi_f : (D^2)^l \times (S^1)^{r-l} \to \pi^{-1}(\square_f) = (S^1)^r \times \square_f / \sim_\Lambda$$

and

$$X^0 \hookrightarrow \cdots \hookrightarrow X^r \hookrightarrow \cdots \hookrightarrow X^{r+n} = M(P^n, \lambda).$$

The (co)homology of the pair (X^{r+l}, X^{r+l-1})

There is an isomorphism

$$\bigoplus_{\dim f = n-l} \Phi_{f*} : \bigoplus_{\dim f = n-l} H_p\big((D^2)^l \times (S^1)^{r-l}, \partial(D^2)^l \times (S^1)^{r-l}\big) \xrightarrow{\cong} H_p\big(X^{r+l}, X^{r+l-1}\big), \quad \forall p \geq 0$$

Thus we have the description

$$H_j(X^{r+l}, X^{r+l-1}) \cong H^j(X^{r+l}, X^{r+l-1}; \mathbb{Z}) \cong \begin{cases} \bigoplus_{\dim f = n-l} \mathbb{Z}^{\binom{r-l}{j-2l}}, & \text{when } 2l \leq j \leq r+l \\ 0, & \text{when } j < 2l \text{ or } j > r+l \end{cases}$$

The cohomology of quotients of moment-angle manifolds over polygons

Betti numbers of $\mathcal{Z}_{P^2}$

Consider the special case of the quotient: $\mathcal{Z}_{P^2} = M(P^2, id)$ itself, i.e., $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^m$ is trivial. Set

$$\begin{cases} \Lambda_{F_1} = (e_1, \cdots, e_m), \ldots, \Lambda_{F_{m-1}} = (e_{m-1}, e_1, \cdots, e_{m-2}, e_m), \Lambda_{F_m} = (e_m, e_1, \cdots, e_{m-1}); \\ \Lambda_{F_1 \cap F_2} = (e_1, e_2, e_3, \cdots, e_m), \ldots, \Lambda_{F_{m-1} \cap F_m} = (e_{m-1}, e_m, e_1, \cdots, e_{m-2}), \Lambda_{F_1 \cap F_m} = (e_1, e_m, e_2, \cdots, e_{m-1}). \end{cases}$$

Then the filtration $X^0 \hookrightarrow \cdots \hookrightarrow X^m \hookrightarrow X^{m+1} \hookrightarrow X^{m+2} = \mathcal{Z}_{P^2}$ induce a spectral sequence which has the E^1-page:

$$0 \leftarrow E^1_{m+2,m+2} \xleftarrow{\delta} E^1_{m+1,m+1} \xleftarrow{\delta} E^1_{m,m} \xleftarrow{0} \cdots \xleftarrow{0} E^1_{4,4} \xleftarrow{0} E^1_{3,3} \xleftarrow{0} E^1_{2,2} \xleftarrow{0} E^1_{1,1} \xleftarrow{0} E^1_{0,0} \leftarrow 0$$

$$0 \qquad \leftarrow E^1_{m+1,m+2} \xleftarrow{\delta} E^1_{m,m+1} \leftarrow 0$$

$$\ddots$$

$$0 \leftarrow E^1_{4,m+2} \xleftarrow{\delta} E^1_{3,m+1} \leftarrow 0$$

$$0 \leftarrow 0 \leftarrow E^1_{2,m+1} \leftarrow 0$$

where, $E^1_{*,m+l} = H^*\left(X^{m+l}, X^{m+l-1}; \mathbb{Z}\right)$.

The cohomology of quotients of moment-angle manifolds over polygons

Betti numbers of $\mathcal{Z}_{P^2}$

Consider the special case of the quotient: $\mathcal{Z}_{P^2} = M(P^2, id)$ itself, i.e., $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^m$ is trivial. Set

$$\begin{cases} \Lambda_{F_1} = (e_1, \cdots, e_m), \ldots, \Lambda_{F_{m-1}} = (e_{m-1}, e_1, \cdots, e_{m-2}, e_m), \Lambda_{F_m} = (e_m, e_1, \cdots, e_{m-1}); \\ \Lambda_{F_1 \cap F_2} = (e_1, e_2, e_3, \cdots, e_m), \ldots, \Lambda_{F_{m-1} \cap F_m} = (e_{m-1}, e_m, e_1, \cdots, e_{m-2}), \Lambda_{F_1 \cap F_m} = (e_1, e_m, e_2, \cdots, e_{m-1}). \end{cases}$$

Then the filtration $X^0 \hookrightarrow \cdots \hookrightarrow X^m \hookrightarrow X^{m+1} \hookrightarrow X^{m+2} = \mathcal{Z}_{P^2}$ induce a spectral sequence which has the E^1-page:

$$\begin{array}{cccccccccc}
0 \leftarrow E^1_{m+2,m+2} \xleftarrow{\delta} E^1_{m+1,m+1} \xleftarrow{\delta} E^1_{m,m} & \xleftarrow{0} \cdots \xleftarrow{0} E^1_{4,4} & \xleftarrow{0} E^1_{3,3} & \xleftarrow{0} E^1_{2,2} & \xleftarrow{0} E^1_{1,1} \xleftarrow{0} E^1_{0,0} \leftarrow 0 \\
0 \qquad \leftarrow E^1_{m+1,m+2} \xleftarrow{\delta} E^1_{m,m+1} \leftarrow 0 & & & & \\
& \ddots & & & \\
& 0 \quad \leftarrow E^1_{4,m+2} & \xleftarrow{\delta} E^1_{3,m+1} \leftarrow 0 & & \\
& & 0 \quad \leftarrow 0 & \leftarrow E^1_{2,m+1} \leftarrow 0 &
\end{array}$$

where, $E^1_{*,m+l} = H^*\left(X^{m+l}, X^{m+l-1}; \mathbb{Z}\right)$.

The cohomology of quotients of moment-angle manifolds over polygons

Betti numbers of $\mathcal{Z}_{P^2}$

And the E^2- to E^m-page:

$$
\begin{tikzcd}
E^2_{m+2,m+2} & 0 & 0 & E^1_{m-1,m-1} \arrow[dl, "\delta_2"] & \cdots & E^1_{4,4} \arrow[ddl, "\delta_{m-3}"] & E^1_{3,3} \arrow[dddl, "\delta_{m-2}"] & E^1_{2,2} \arrow[ddddl, "\delta_{m-1}"] & E^1_{1,1} \arrow[dddddl, "\delta_m"] & E^1_{0,0} \\
& 0 & E^2_{m,m+1} & 0 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & 0 & E^2_{4,m+1} & 0 & & & \\
& & & & & 0 & E^2_{3,m+1} & 0 & & \\
& & & & & & 0 & E^1_{2,m+1} & 0 &
\end{tikzcd}
$$

Thus we will finish an another proof of which was obtained by considering the E_2 term of the Leray-Serre spectral sequence for the fibre bundle $T^{m-2} \to \mathcal{Z}_{P^2} \to M^4$:

[Buchstaber and Panov, 1999]

$$
\dim H^p(\mathcal{Z}_{P^2};\mathbb{Z}) = \begin{cases} (m-2)\binom{m-2}{p-2} - \binom{m-2}{p-1} - \binom{m-2}{p-3}, & \text{if } 3 \le p \le m-1; \\ 0, \ \text{if } p = 1, 2, m, \text{or } m+1; & 1, \ \text{if } p = 0, \text{or } m+2. \end{cases}
$$

The cohomology of quotients of moment-angle manifolds over polygons

Betti numbers of $\mathcal{Z}_{P^2}$

And the E^2- to E^m-page:

$$\begin{tikzcd}
E^2_{m+2,m+2} & 0 & 0 & E^1_{m-1,m-1} \arrow[dl, "\delta_2"'] & \cdots & E^1_{4,4} \arrow[ddl, "\delta_{m-3}"] & E^1_{3,3} \arrow[dddl, "\delta_{m-2}"] & E^1_{2,2} \arrow[ddddl, "\delta_{m-1}"] & E^1_{1,1} \arrow[dddddl, "\delta_m"] & E^1_{0,0} \\
 & 0 & E^2_{m,m+1} & 0 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & 0 & E^2_{4,m+1} & 0 & & & \\
 & & & & & 0 & E^2_{3,m+1} & 0 & & \\
 & & & & & & 0 & E^1_{2,m+1} & 0 &
\end{tikzcd}$$

Thus we will finish an another proof of which was obtained by considering the E_2 term of the Leray-Serre spectral sequence for the fibre bundle $T^{m-2} \to \mathcal{Z}_{P^2} \to M^4$:

[Buchstaber and Panov, 1999]

$$\dim H^p(\mathcal{Z}_{P^2};\mathbb{Z}) = \begin{cases} (m-2)\binom{m-2}{p-2} - \binom{m-2}{p-1} - \binom{m-2}{p-3}, \text{ if } 3 \le p \le m-1; \\ 0, \text{ if } p = 1, 2, m, \text{or } m+1; \quad 1, \text{ if } p = 0, \text{or } m+2. \end{cases}$$

The cohomology of quotients of moment-angle manifolds over polygons

The cohomology ring of $\mathcal{Z}_{P2}$

Also in [Buchstaber and Panov, 1999 and 2002 (Proposition 7.23)], they gave a description of the cohomology ring of $\mathcal{Z}_{P2}$: the group $H^{m+2}(\mathcal{Z}_{P2})$ is one-dimensional with the fundamental class $[v_1 v_2 u_3 \cdots u_m]$ of the manifold as the generator, the product of two cohomology classes $[v_{i_1} u_{\tau_1}] \in H^{2+|\tau_1|}(\mathcal{Z}_{P2})$ and $[v_{i_2} u_{\tau_2}] \in H^{2+|\tau_2|}(\mathcal{Z}_{P2})$ equals $[v_1 v_2 u_3 \cdots u_m]$ (**up to a sign**) only when $\{\{i_1\}, \{i_2\}, \tau_1, \tau_2\}$ is a partition of $[m]$, and otherwise equals zero. Motivated by this, using a filtration-preserving approximation of the diagonal map,

An explicit presentation of $H^*(\mathcal{Z}_{P2};\mathbb{Z})$

$$H^*(\mathcal{Z}_{P2};\mathbb{Z}) \cong \bigwedge[e, \{e_{\mathcal{Q},l_a}\}_{\mathcal{Q},a}] \,/\, \mathcal{R}$$

where e is a generator of $H^{m+2}(\mathcal{Z}_{P2})$, and $e_{\mathcal{Q},l_a}$ is a generator of $H^{1+|\mathcal{Q}|}(\mathcal{Z}_{P2})$. Let (*) denote the complementary condition of $(\mathcal{Q}, \mathcal{Q}')$, the ideal $\mathcal{R}$ is generated by the relations

$$e_{\mathcal{Q},l_a} e_{\mathcal{Q}',l'_{a'}} = \begin{cases} 0, & \text{if } (\mathcal{Q},\mathcal{Q}') \text{ does not satisfy (*)} \\ 0, & \text{if } (\mathcal{Q},\mathcal{Q}') \text{ satisfies (*) but } a > a' \text{ or } a+1 < a' \\ (-1)^{l_a - l_{a-1}} e, & \text{if } (\mathcal{Q},\mathcal{Q}') \text{ satisfies (*) and } a = a' \\ (-1)^{l'_{a+1} - l'_a} e, & \text{if } (\mathcal{Q},\mathcal{Q}') \text{ satisfies (*) and } a+1 = a' \end{cases}$$

The cohomology of quotients of moment-angle manifolds over polygons

The cohomology ring of $\mathcal{Z}_{p2}$

Also in [Buchstaber and Panov, 1999 and 2002 (Proposition 7.23)], they gave a description of the cohomology ring of $\mathcal{Z}_{P^2}$: the group $H^{m+2}(\mathcal{Z}_{P^2})$ is one-dimensional with the fundamental class $[v_1 v_2 u_3 \cdots u_m]$ of the manifold as the generator, the product of two cohomology classes $[v_{i_1} u_{\tau_1}] \in H^{2+|\tau_1|}(\mathcal{Z}_{P^2})$ and $[v_{i_2} u_{\tau_2}] \in H^{2+|\tau_2|}(\mathcal{Z}_{P^2})$ equals $[v_1 v_2 u_3 \cdots u_m]$ (**up to a sign**) only when $\{\{i_1\}, \{i_2\}, \tau_1, \tau_2\}$ is a partition of $[m]$, and otherwise equals zero. Motivated by this, using a filtration-preserving approximation of the diagonal map,

An explicit presentation of $H^*(\mathcal{Z}_{p2};\mathbb{Z})$

$$H^*(\mathcal{Z}_{P^2};\mathbb{Z}) \cong \bigwedge[e, \{e_{\mathcal{Q},l_a}\}_{\mathcal{Q},a}] \;/\; \mathcal{R}$$

where e is a generator of $H^{m+2}(\mathcal{Z}_{P^2})$, and $e_{\mathcal{Q},l_a}$ is a generator of $H^{1+|\mathcal{Q}|}(\mathcal{Z}_{P^2})$. Let (*) denote the complementary condition of $(\mathcal{Q}, \mathcal{Q}')$, the ideal $\mathcal{R}$ is generated by the relations

$$e_{\mathcal{Q},l_a} e_{\mathcal{Q}',l'_{a'}} = \begin{cases} 0, & \text{if } (\mathcal{Q}, \mathcal{Q}') \text{ does not satisfy (*)} \\ 0, & \text{if } (\mathcal{Q}, \mathcal{Q}') \text{ satisfies (*) but } a > a' \text{ or } a+1 < a' \\ (-1)^{l_a - l_{a-1}} e, & \text{if } (\mathcal{Q}, \mathcal{Q}') \text{ satisfies (*) and } a = a' \\ (-1)^{l'_{a+1} - l'_a} e, & \text{if } (\mathcal{Q}, \mathcal{Q}') \text{ satisfies (*) and } a+1 = a' \end{cases}$$

The cohomology of quotients of moment-angle manifolds over polygons

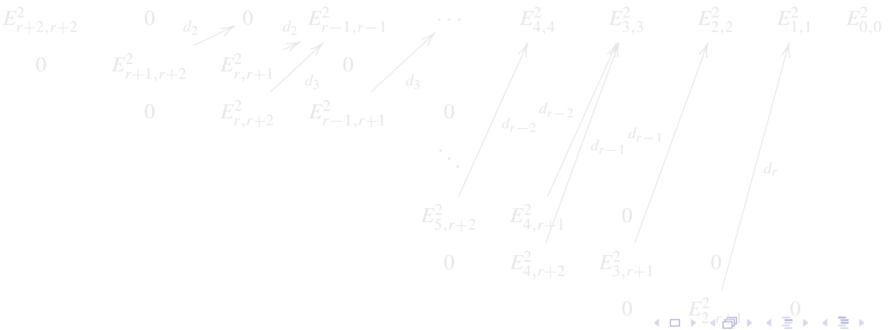
(Co)homology groups of $M(P^2, \Lambda)$ under a special class of colourings

What else is this filtration able to do? Given $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^r$ such that all the 3 vectors $\{\Lambda(e_m), \Lambda(e_1), \Lambda(e_2)\}, \ldots, \{\Lambda(e_{m-1}), \Lambda(e_m), \Lambda(e_1)\}$ from adjacent edges are dependent. Set

$$\begin{cases} \Lambda_{F_1 \cap F_2} = (\Lambda_1, \Lambda_2, \alpha_1, \cdots, \alpha_{r-2}), \cdots, \\ \Lambda_{F_{m-1} \cap F_m} = (\Lambda_{m-1}, \Lambda_m, \alpha_1, \cdots, \alpha_{r-2}), \Lambda_{F_1 \cap F_m} = (\Lambda_1, \Lambda_m, \alpha_1, \cdots, \alpha_{r-2}); \\ \Lambda_{F_1} = \Lambda_{F_1 \cap F_2}, \Lambda_{F_2} = \Lambda_{F_2 \cap F_3}, \cdots, \Lambda_{F_m} = (\Lambda_m, \Lambda_1, \alpha_1, \cdots, \alpha_{r-2}) \end{cases}$$

where we choose $\alpha_1, \cdots, \alpha_{r-2} \in \mathbb{Z}^r$ such that $|\det \Lambda_{F_1 \cap F_2}| = \cdots = |\det \Lambda_{F_1 \cap F_m}| = 1$.

Then the induced spectral sequence has the same form of E^1-page as before, and the E^2- to E^r-page as:



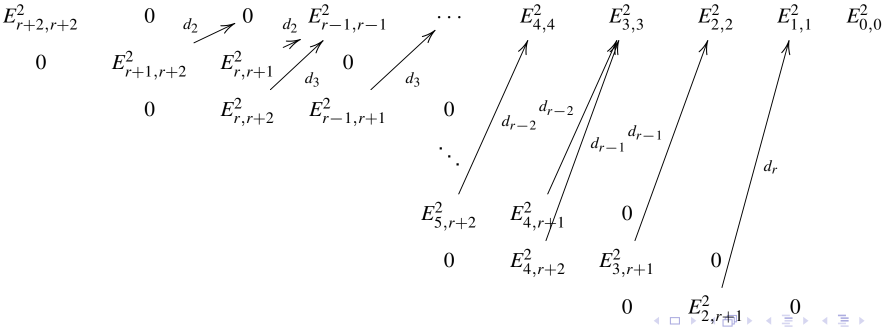
The cohomology of quotients of moment-angle manifolds over polygons

(Co)homology groups of $M(P^2, \Lambda)$ under a special class of colourings

What else is this filtration able to do? Given $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^r$ such that all the 3 vectors $\{\Lambda(e_m), \Lambda(e_1), \Lambda(e_2)\}, \ldots, \{\Lambda(e_{m-1}), \Lambda(e_m), \Lambda(e_1)\}$ from adjacent edges are dependent. Set

$$\begin{cases} \Lambda_{F_1 \cap F_2} = (\Lambda_1, \Lambda_2, \alpha_1, \cdots, \alpha_{r-2}), \cdots, \\ \Lambda_{F_{m-1} \cap F_m} = (\Lambda_{m-1}, \Lambda_m, \alpha_1, \cdots, \alpha_{r-2}), \Lambda_{F_1 \cap F_m} = (\Lambda_1, \Lambda_m, \alpha_1, \cdots, \alpha_{r-2}); \\ \Lambda_{F_1} = \Lambda_{F_1 \cap F_2}, \Lambda_{F_2} = \Lambda_{F_2 \cap F_3}, \cdots, \Lambda_{F_m} = (\Lambda_m, \Lambda_1, \alpha_1, \cdots, \alpha_{r-2}) \end{cases}$$

where we choose $\alpha_1, \cdots, \alpha_{r-2} \in \mathbb{Z}^r$ such that $|\det \Lambda_{F_1 \cap F_2}| = \cdots = |\det \Lambda_{F_1 \cap F_m}| = 1$.
Then the induced spectral sequence has the same form of E^1-page as before, and the E^2- to E^r-page as:



The cohomology of quotients of moment-angle manifolds over polygons

(Co)homology groups of $M(P^2, \Lambda)$ under a special class of colourings

Betti numbers of $M(P^2, \Lambda)$

$$H_p(M(P^2, \lambda)) \cong \mathbb{Z}^{\binom{r-2}{p-4} + (m-2)\cdot\binom{r-2}{p-2} + \binom{r-2}{p}}, \text{ for } 0 \le p \le r+2$$

note that we define the binomial $\binom{a}{b}$ has the property that

$$\binom{a}{b} = 0, \text{ if } a < b \text{ or } b < 0.$$

Remark

Apparently it would be interesting if the Betti numbers of all the quotients of moment-angle manifolds over polygons were given, however for general situations the differentials d would be too complicated to deal with, it would be hard to represent their kernels and images, nevertheless there would be some hope to achive this goal with this spectral sequence if we could find some way to simplify the calculation.

The cohomology of quotients of moment-angle manifolds over polygons

(Co)homology groups of $M(P^2, \Lambda)$ under a special class of colourings

Betti numbers of $M(P^2, \Lambda)$

$$H_p(M(P^2, \lambda)) \cong \mathbb{Z}^{\binom{r-2}{p-4}+(m-2)\cdot\binom{r-2}{p-2}+\binom{r-2}{p}}, \text{ for } 0 \leq p \leq r+2$$

note that we define the binomial $\binom{a}{b}$ has the property that

$$\binom{a}{b} = 0, \text{ if } a < b \text{ or } b < 0.$$

Remark

Apparently it would be interesting if the Betti numbers of all the quotients of moment-angle manifolds over polygons were given, however for general situations the differentials d would be too complicated to deal with, it would be hard to represent their kernels and images, nevertheless there would be some hope to achive this goal with this spectral sequence if we could find some way to simplify the calculation.

Fundamental groups of real toric spaces over simple polytopes

Let $\mathcal{F}(P)$ be the set of facets,

Definition: the right-angled Coxeter group [*Combinatorics of Coxeter Groups*, Bjorner and Brenti]

$$W_P =< s_F : F \in \mathcal{F}(P) | s_F^2 = 1; (s_F s_{F'})^2 = 1 \text{ if } F \cap F' \neq \emptyset >,$$

note that $s_F s_{F'}$ has infinite order in W_P if $F \cap F' = \emptyset$.

[Davis and Januszkiewicz, 1991]

The Borel construction of the small cover M_λ, has the fundamental group

$$\pi_1(E(\mathbb{Z}/2)^n \times_{(\mathbb{Z}/2)^n} M_\lambda) \cong W_P.$$

Besides, the Borel construction gives the fibration $M_\lambda \to E(\mathbb{Z}/2)^n \times_{(\mathbb{Z}/2)^n} M_\lambda \to B(\mathbb{Z}/2)^n$, which induces the short exact sequence

$$1 \to \pi_1(M_\lambda) \to W_P \xrightarrow{\phi} (\mathbb{Z}/2)^n \to 1$$

where $\phi(s_F) = \lambda(F)$.

Indeed, this is also the case for real toric spaces M_λ over simple polytopes, the fibration given by the Borel construction also induces the exact sequence

$$1 \to \pi_1(M_\lambda) \to W_P \xrightarrow{\phi} (\mathbb{Z}/2)^r \to 1$$

Fundamental groups of real toric spaces over simple polytopes

Let $\mathcal{F}(P)$ be the set of facets,

Definition: the right-angled Coxeter group [*Combinatorics of Coxeter Groups*, Bjorner and Brenti]

$$W_P =< s_F : F \in \mathcal{F}(P) | s_F^2 = 1; (s_F s_{F'})^2 = 1 \text{ if } F \cap F' \neq \emptyset >,$$

note that $s_F s_{F'}$ has infinite order in W_P if $F \cap F' = \emptyset$.

[Davis and Januszkiewicz, 1991]

The Borel construction of the small cover M_λ, has the fundamental group

$$\pi_1(E(\mathbb{Z}/2)^n \times_{(\mathbb{Z}/2)^n} M_\lambda) \cong W_P.$$

Besides, the Borel construction gives the fibration $M_\lambda \to E(\mathbb{Z}/2)^n \times_{(\mathbb{Z}/2)^n} M_\lambda \to B(\mathbb{Z}/2)^n$, which induces the short exact sequence

$$1 \to \pi_1(M_\lambda) \to W_P \xrightarrow{\phi} (\mathbb{Z}/2)^n \to 1$$

where $\phi(s_F) = \lambda(F)$.

Indeed, this is also the case for real toric spaces M_λ over simple polytopes, the fibration given by the Borel construction also induces the exact sequence

$$1 \to \pi_1(M_\lambda) \to W_P \xrightarrow{\phi} (\mathbb{Z}/2)^r \to 1$$

Fundamental groups of real toric spaces over simple polytopes

However we still would like to study the fundamental group of the real toric space with some other methods, which will lead to an explicit presentation of $\pi_1(M_\lambda)$:

When M_λ is a small cover [Wu and Yu, 2021]

$$\pi_1(M_\lambda) \cong \left\langle \beta_{F_i}^{(g)} : g \in (\mathbb{Z}/2)^n, i = 1, \dots, m \;\middle|\; \begin{array}{r} (\beta_{F_i}^{(g)})^{-1} = \beta_{F_i}^{(g+\lambda(F_i))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^n; \\ \beta_{F_i}^{(g)} \cdot \beta_{F_j}^{(g+\lambda(F_i))} = \beta_{F_j}^{(g)} \cdot \beta_{F_i}^{(g+\lambda(F_j))}, \text{ whenever } F_i \cap F_j \neq \emptyset \end{array} \right\rangle$$

Rewiew: cellular decomposition of M_λ

Let $\Theta : (\mathbb{Z}/2)^r \times P^n \to M_\lambda$, $G_f =$Span$\{\lambda(e_{i_1}), \dots, \lambda(e_{i_l})\}$, and $C_f^{(g)} = \bigcup_{h \in g+G_f} \Theta(h \times \square_f)$. Then $C_f^{(g)} \approx [-1,1]^{n-\dim f}$, which is the form of an l-cell. Note that

$$C_f^{(g)} = \cdots = C_f^{(g+\epsilon_1\lambda(F_{i_1})+\cdots+\epsilon_l\lambda(F_{i_l}))} = \cdots = C_f^{(g+\lambda(F_{i_1})+\cdots+\lambda(F_{i_l}))}, \ \forall \epsilon_1, \dots, \epsilon_l \in \{0,1\}$$

Specially, there are 2^r 0-cells $\{C_P^{(0,\cdots,0)}, \dots, C_P^{(1,\cdots,1)}\}$, $m \cdot 2^{r-1}$ 1-cells $\{C_{F_i}^{(g)} : i = 1, \dots, m, \ g \in (\mathbb{Z}/2)^r\}$, $f_{n-2} \cdot 2^{r-2}$ 2-cells $\{C_{F \cap F'}^{(g)} : F \cap F' \neq \emptyset, \ g \in (\mathbb{Z}/2)^r\}$, and $f_0 \cdot 2^{r-n}$ n-cells $\{C_v^{(g)} : v$ is a vertex of $P^n, \ g \in (\mathbb{Z}/2)^r\}$, note that if $g - h \notin G_v$, $C_v^{(g)} \cap C_v^{(h)} = \emptyset$.

Fundamental groups of real toric spaces over simple polytopes

However we still would like to study the fundamental group of the real toric space with some other methods, which will lead to an explicit presentation of $\pi_1(M_\lambda)$:

When M_λ is a small cover [Wu and Yu, 2021]

$$\pi_1(M_\lambda) \cong \left\langle \beta_{F_i}^{(g)} : g \in (\mathbb{Z}/2)^n, i = 1, \ldots, m \;\middle|\; \begin{array}{c} (\beta_{F_i}^{(g)})^{-1} = \beta_{F_i}^{(g+\lambda(F_i))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^n; \\ \beta_{F_i}^{(g)} \cdot \beta_{F_j}^{(g+\lambda(F_i))} = \beta_{F_j}^{(g)} \cdot \beta_{F_i}^{(g+\lambda(F_j))}, \text{ whenever } F_i \cap F_j \neq \emptyset \end{array} \right\rangle$$

Rewiew: cellular decomposition of M_λ

Let $\Theta : (\mathbb{Z}/2)^r \times P^n \to M_\lambda$, $G_f =$Span$\{\lambda(e_{i_1}), \ldots, \lambda(e_{i_l})\}$, and $C_f^{(g)} = \bigcup_{h \in g+G_f} \Theta(h \times \Box_f)$. Then $C_f^{(g)} \approx [-1,1]^{n-\dim f}$, which is the form of an l-cell. Note that

$$C_f^{(g)} = \cdots = C_f^{(g+\epsilon_1\lambda(F_{i_1})+\cdots+\epsilon_l\lambda(F_{i_l}))} = \cdots = C_f^{(g+\lambda(F_{i_1})+\cdots+\lambda(F_{i_l}))}, \ \forall \epsilon_1, \ldots, \epsilon_l \in \{0,1\}$$

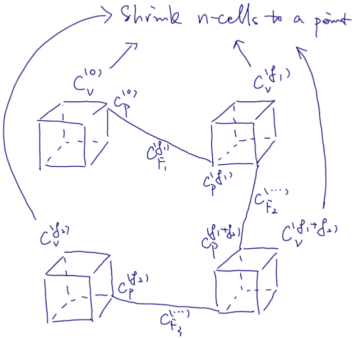
Specially, there are 2^r 0-cells $\{C_P^{(0,\cdots,0)}, \ldots, C_P^{(1,\cdots,1)}\}$, $m \cdot 2^{r-1}$ 1-cells $\{C_{F_i}^{(g)} : i = 1, \ldots, m, \ g \in (\mathbb{Z}/2)^r\}$, $f_{n-2} \cdot 2^{r-2}$ 2-cells $\{C_{F \cap F'}^{(g)} : F \cap F' \neq \emptyset, \ g \in (\mathbb{Z}/2)^r\}$, and $f_0 \cdot 2^{r-n}$ n-cells $\{C_v^{(g)} : v$ is a vertex of P^n, $g \in (\mathbb{Z}/2)^r\}$, note that if $g - h \notin G_v$, $C_v^{(g)} \cap C_v^{(h)} = \emptyset$.

Fundamental groups of real toric spaces over simple polytopes

As there exists 2^r 0-cells in the CW-complex M_λ, it is hard to calculate the fundamental group directly. Thus we need to do some deformation:

The first time of retract

There exists a CW-complex M', which is homotopically equivalent to the original manifold M_λ, has 2^{r-n} 0-cells, and $(f_{n-l} - \binom{n}{l}) \cdot 2^{r-l}$ l-cells for $l = 1, \ldots, n$.



Fundamental groups of real toric spaces over simple polytopes

The second time of retract

There exists a CW-complex $\tilde{M}$, which is homotopically equivalent to the original manifold M_λ, has exactly only one 0-cell, $(m-n)\cdot 2^{r-1} - 2^{r-n} + 1$ 1-cells, and $(f_{n-l} - \binom{n}{l}) \cdot 2^{r-l}$ l-cells for $l = 2, \ldots, n$.

We consider the 1-skeleton of M' as a graph, the vertices and edges of which are the 0-cells and 1-cells of M' respectively. We obtain the needed CW-complex $\tilde{M}$ by extracting a maximal tree out of this graph.

Remark

In the special case when $r = n$, i.e. M_λ is a small cover, M' has only one 0-cell, thus the 1-skeleton is just a finite wedge of circles.

The fundamental group of the 1-skeleton $\tilde{M}^1$ can be written as

$$\pi_1(\tilde{M}^1, D_v) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(m-n)\cdot 2^{r-1} - 2^{r-n}+1}$$

$$\cong \left\langle \beta_F^{(g)} : g \in (\mathbb{Z}/2)^r, F \in \mathcal{F}(P) \;\middle|\; \begin{array}{c} (\beta_F^{(g)})^{-1} = \beta_F^{(g+\lambda(F))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^r; \\ \beta_{F_{n+1}}^{(0)} = \cdots = \beta_{F_r}^{(0)} = \cdots = \beta_{F_r}^{(\lambda(F_{n+1}) + \cdots + \lambda(F_{r-1}))} = 1 \end{array} \right\rangle$$

Fundamental groups of real toric spaces over simple polytopes

The second time of retract

There exists a CW-complex $\tilde{M}$, which is homotopically equivalent to the original manifold M_λ, has exactly only one 0-cell, $(m-n)\cdot 2^{r-1} - 2^{r-n} + 1$ 1-cells, and $(f_{n-l} - \binom{n}{l}) \cdot 2^{r-l}$ l-cells for $l = 2, \ldots, n$.

We consider the 1-skeleton of M' as a graph, the vertices and edges of which are the 0-cells and 1-cells of M' respectively. We obtain the needed CW-complex $\tilde{M}$ by extracting a maximal tree out of this graph.

Remark

In the special case when $r = n$, i.e. M_λ is a small cover, M' has only one 0-cell, thus the 1-skeleton is just a finite wedge of circles.

The fundamental group of the 1-skeleton $\tilde{M}^1$ can be written as

$$\pi_1(\tilde{M}^1, D_v) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(m-n)\cdot 2^{r-1} - 2^{r-n}+1}$$

$$\cong \left\langle \beta_F^{(g)} : g \in (\mathbb{Z}/2)^r, F \in \mathcal{F}(P) \;\middle|\; \begin{array}{c} (\beta_F^{(g)})^{-1} = \beta_F^{(g+\lambda(F))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^r; \\ \beta_{F_{n+1}}^{(0)} = \cdots = \beta_{F_r}^{(0)} = \cdots = \beta_{F_r}^{(\lambda(F_{n+1}) + \cdots + \lambda(F_{r-1}))} = 1 \end{array} \right\rangle$$

Fundamental groups of real toric spaces over simple polytopes

The second time of retract

There exists a CW-complex $\tilde{M}$, which is homotopically equivalent to the original manifold M_λ, has exactly only one 0-cell, $(m-n)\cdot 2^{r-1} - 2^{r-n} + 1$ 1-cells, and $(f_{n-l} - \binom{n}{l}) \cdot 2^{r-l}$ l-cells for $l = 2, \ldots, n$.

We consider the 1-skeleton of M' as a graph, the vertices and edges of which are the 0-cells and 1-cells of M' respectively. We obtain the needed CW-complex $\tilde{M}$ by extracting a maximal tree out of this graph.

Remark

In the special case when $r = n$, i.e. M_λ is a small cover, M' has only one 0-cell, thus the 1-skeleton is just a finite wedge of circles.

The fundamental group of the 1-skeleton $\tilde{M}^1$ can be written as

$$\pi_1(\tilde{M}^1, D_v) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(m-n)\cdot 2^{r-1} - 2^{r-n}+1}$$

$$\cong \left\langle \beta_F^{(g)} : g \in (\mathbb{Z}/2)^r, F \in \mathcal{F}(P) \;\middle|\; \begin{array}{c} (\beta_F^{(g)})^{-1} = \beta_F^{(g+\lambda(F))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^r; \\ \beta_{F_{n+1}}^{(0)} = \cdots = \beta_{F_r}^{(0)} = \cdots = \beta_{F_r}^{(\lambda(F_{n+1}) + \cdots + \lambda(F_{r-1}))} = 1 \end{array} \right\rangle$$

Fundamental groups of real toric spaces over simple polytopes

The second time of retract

There exists a CW-complex $\tilde{M}$, which is homotopically equivalent to the original manifold M_λ, has exactly only one 0-cell, $(m-n)\cdot 2^{r-1} - 2^{r-n} + 1$ 1-cells, and $(f_{n-l} - \binom{n}{l}) \cdot 2^{r-l}$ l-cells for $l = 2, \ldots, n$.

We consider the 1-skeleton of M' as a graph, the vertices and edges of which are the 0-cells and 1-cells of M' respectively. We obtain the needed CW-complex $\tilde{M}$ by extracting a maximal tree out of this graph.

Remark

In the special case when $r = n$, i.e. M_λ is a small cover, M' has only one 0-cell, thus the 1-skeleton is just a finite wedge of circles.

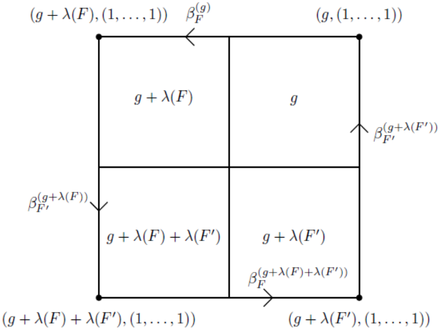
The fundamental group of the 1-skeleton $\tilde{M}^1$ can be written as

$$\pi_1(\tilde{M}^1, D_v) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(m-n)\cdot 2^{r-1} - 2^{r-n} + 1}$$

$$\cong \left\langle \beta_F^{(g)} : g \in (\mathbb{Z}/2)^r, F \in \mathcal{F}(P) \;\middle|\; \begin{array}{c} (\beta_F^{(g)})^{-1} = \beta_F^{(g+\lambda(F))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^r; \\ \beta_{F_{n+1}}^{(0)} = \cdots = \beta_{F_r}^{(0)} = \cdots = \beta_{F_r}^{(\lambda(F_{n+1}) + \cdots + \lambda(F_{r-1}))} = 1 \end{array} \right\rangle$$

Fundamental groups of real toric spaces over simple polytopes

Whenever the given facets $F \cap F' \neq \emptyset$, they will determine a 2-cell of M_λ drawn as:

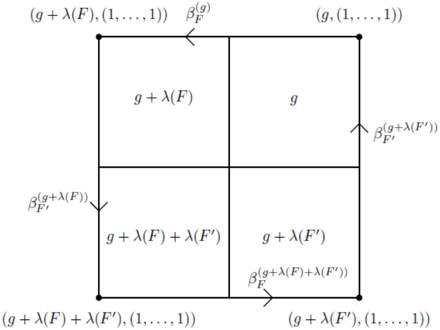


Thus, we obtain the relation set

$$R = \{\beta_F^{(g)} \cdot \beta_{F'}^{(g+\lambda(F))} \cdot \beta_F^{(g+\lambda(F)+\lambda(F'))} \cdot \beta_{F'}^{(g+\lambda(F'))} : F \cap F' \neq \emptyset\} \subset \pi_1(\tilde{M}^1, D_v)$$

Fundamental groups of real toric spaces over simple polytopes

Whenever the given facets $F \cap F' \neq \emptyset$, they will determine a 2-cell of M_λ drawn as:



Thus, we obtain the relation set

$$R = \{\beta_F^{(g)} \cdot \beta_{F'}^{(g+\lambda(F))} \cdot \beta_F^{(g+\lambda(F)+\lambda(F'))} \cdot \beta_{F'}^{(g+\lambda(F'))} : F \cap F' \neq \emptyset\} \subset \pi_1(\tilde{M}^1, D_v)$$

Fundamental groups of real toric spaces over simple polytopes

The fundamental group of M_λ

$$\pi_1(M_\lambda) \cong \pi_1(\tilde{M}, D_v) \cong \pi_1(\tilde{M}^1, D_v)/(\text{normalization of } R)$$

$$\cong \left\langle \beta_{F_i}^{(g)} : g \in (\mathbb{Z}/2)^r, i = 1, \dots, m \;\middle|\; \begin{array}{c} (\beta_{F_i}^{(g)})^{-1} = \beta_{F_i}^{(g+\lambda(F_i))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^r; \\ \beta_{F_{n+1}}^{(0)} = \cdots = \beta_{F_r}^{(0)} = \cdots = \beta_{F_r}^{(\lambda(F_{n+1})+\cdots+\lambda(F_{r-1}))} = 1; \\ \beta_{F_i}^{(g)} \cdot \beta_{F_j}^{(g+\lambda(F_i))} = \beta_{F_j}^{(g)} \cdot \beta_{F_i}^{(g+\lambda(F_j))}, \text{ whenever } F_i \cap F_j \neq \emptyset \end{array} \right\rangle$$

Thank you for your attention!

Fundamental groups of real toric spaces over simple polytopes

The fundamental group of M_λ

$$\pi_1(M_\lambda) \cong \pi_1(\tilde{M}, D_v) \cong \pi_1(\tilde{M}^1, D_v)/(\text{normalization of } R)$$

$$\cong \left\langle \beta_{F_i}^{(g)} : g \in (\mathbb{Z}/2)^r, i = 1, \dots, m \;\middle|\; \begin{array}{c} (\beta_{F_i}^{(g)})^{-1} = \beta_{F_i}^{(g+\lambda(F_i))}; \\ \beta_{F_1}^{(g)} = \cdots = \beta_{F_n}^{(g)} = 1, \quad \forall g \in (\mathbb{Z}/2)^r; \\ \beta_{F_{n+1}}^{(0)} = \cdots = \beta_{F_r}^{(0)} = \cdots = \beta_{F_r}^{(\lambda(F_{n+1})+\cdots+\lambda(F_{r-1}))} = 1; \\ \beta_{F_i}^{(g)} \cdot \beta_{F_j}^{(g+\lambda(F_i))} = \beta_{F_j}^{(g)} \cdot \beta_{F_i}^{(g+\lambda(F_j))}, \text{ whenever } F_i \cap F_j \neq \emptyset \end{array} \right\rangle$$

Thank you for your attention!