Equivariant formality of isospectral matrix manifolds

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(based on joint works with V.Buchstaber, V.Cherepanov, M.Masuda, G.Solomadin, K.Sorokin)

For a simple graph Γ on n vertices, consider a space $M(\Gamma,\lambda)$ of all Γ-shaped Hermitian matrices of size n with a given simple spectrum λ. If a spectrum is generic, then $M(\Gamma,\lambda)$ is smooth, and it carries a canonical torus action with isolated fixed points. Some examples are well studied: if Γ is a path graph, $M(\Gamma,\lambda)$ is "the complex Tomei manifold" - the quasitoric manifold over a permutohedron generated by the proper coloring of its facets. If Γ is a complete graph, $M(\Gamma,\lambda)$ is the full flag variety.

The main problem: which manifolds $M(\Gamma,\lambda)$ are equivariantly formal? We proved the following alternative:

(1) Γ is a proper interval graph, $M(\Gamma,\lambda)$ is equivariantly formal, moreover, it is a twin of a regular semisimple Hessenberg variety. Here we have $\beta(M(\Gamma,\lambda)) = n!$.

(2) Γ is not proper interval, $M(\Gamma,\lambda)$ is not equivariantly formal, and we have $\beta(M(\Gamma,\lambda)) > n!$.

The proof is partly theoretical, partly computational. A similar alternative holds for real symmetric matrices, discrete torus actions, and cohomology with $\mathbb{Z}/2$ coefficients. These alternatives can be interpreted in terms of applied algebra: staircase matrices can be asymptotically diagonalized (e.g. by the QR-algorithm), while it is impossible to diagonalize non-staircase matrices inside their sparsity class. The proven result may be applied to make this statement meaningful.