Cohomology of moment-angle complex quotients and spectral sequences

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Motivation and plan

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Consider the group action on a moment-angle complex by any closed subgroup in the naturally acting torus on MAC. The orbit space has the quotient torus (residual) action. Study singular cohomology groups and equivariant cohomology rings (e.g. torsion or Betti numbers) of this orbit space using spectral sequences.

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Plan of the talk

0. Introduction: known results on cohomology groups of quotients.
1. Homotopy decomposition for any MAC quotient and its Borel space;
2. Bousfield-Kan spectral sequence collapse at page 2 for every toric $(\mathrm{cat}\, K)$-diagram and its classifying space;
3. Results on BKSS page 2 using sheaf cohomology.

Part 0: Introduction

Moment-angle complexes

For the pair (X, A) of topological spaces the I-product of (X, A) is defined by

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The natural inclusions $(X, A)^I \to (X, A)^J$ for any $I \subseteq J \subseteq [m]$ form a diagram, i.e. a functor $D \in \mathrm{Top}^{\mathrm{cat}\, K}$ over the small category $\mathrm{cat}\, K$ (objects are simplices of K on $[m]$ and $\varnothing$, arrows are inclusions).

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Definition (Buchstaber & Panov '98)

Complex (real) moment-angle complex $\mathcal{Z}_K$ (resp. $\mathcal{R}_K$) a.k.a. polyhedral product for (D^d, S^{d-1}) for $D^d = \{z \in \mathbb{F}_d \mid |z| \leqslant 1\}$, is

$$(D^d, S^{d-1})^K := \operatorname*{colim}_{I\in \mathrm{cat}\,K} (D^d, S^{d-1})^I = \bigcup_{I\in \mathrm{cat}\,K} (D^d, S^{d-1})^I,$$

$$G_d := \begin{cases} \mathbb{Z}/2, & d=1,\\ S^1, & d=2,\end{cases} \qquad \mathbb{F}_d := \begin{cases} \mathbb{R}, & d=1,\\ \mathbb{C}, & d=2.\end{cases}$$

Torus actions on $(D^d, S^{d-1})^K$

- The action $G_d \curvearrowright D^d$, $t \circ z := t \cdot z$, preserves $S^{d-1} = \partial D^d$ and induces $G_d^m \curvearrowright (D^d)^m$, $d = 1, 2$.

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- The quotient $(D^d, S^{d-1})^K / H_d = \operatorname{colim}_{I \in \operatorname{cat} K} (D^d, S^{d-1})^I / H_d$ has the natural residual action of $L_d := G_d^m / H_d$.

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- Contractible space $|\operatorname{cone} K|$ for $H_d = G_d^m$.

Retrospect: cohomology rings of complex MAC quotients

- Hohster '77: (restriction $K_J := \{I \in K|\ I \subseteq J\}$ of K to J)

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- Buchstaber and Panov '98, '02; Panov '99, '15: Tor-algebra description for the cohomology ring of any partial quotient by a subtorus using Eilenberg-Maclane SS (incomplete argument);

$$H^{2j-i}(X_{K,H}; \mathbb{Z}) \cong Tor_{H^*(BL)}^{-i,2j}(\mathbb{Z}[K], \mathbb{Z});$$

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- M. Franz '21: $\cup$-product in cohomology of any partial quotient (twisted product on the Tor-module).

Quotients corresponding to colorings

Let $\lambda\colon [m] \to [k]$ be a map. Denote by $S^1_{i,j}$ the diagonal circle in $S^1_i \times S^1_j$ and let

$$H_\lambda := \prod_{\lambda(i)=\lambda(j)} S^1_{i,j} \subset T^m.$$

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Theorem (Yu Li '19)

$$H^n(\mathcal{Z}_K/H_\lambda;\ \mathbb{Z}) \simeq \bigoplus_{J\subseteq[k]} \widetilde{H}^{n-|J|-1}(K_{\lambda^{-1}(J)};\ \mathbb{Z}).$$

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Theorem (Xin Fu '19)

For any $0\leqslant k\leqslant m-2$ one has

$$\mathcal{Z}_{\Delta^k_m}/S^1_d \simeq \mathbb{C}P^{k+1} \vee \mathcal{Z}_{\Delta^k_{m-1}} \vee \Big(\bigvee_{i=1}^{k} S^{2i-1} * \mathcal{Z}_{\Delta^{k-i}_{m-i-1}}\Big) \vee (S^{2k+1} * T^{m-k-2}).$$

Stabilizers of subgroup and residual actions

Consider the G_d^m-invariant constructible subset

$$Z(I) := \{z = (z_1, \dots, z_m) \in (D^d, S^{d-1})^K \mid z_i = 0 \Leftrightarrow i \in I\} \subseteq (D^d, S^{d-1})^K.$$

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Proposition

1. $\mathrm{Stab}_{H_d}\, x=H_d\cap G_d^I,\ x\in Z(I)$;
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Stabilizers and orbits are not isomorphic at different points of $(D^d,S^{d-1})^I$ (and of $(D^d,S^{d-1})^I/H_d$), in general.

Diagrams S_d, Q_d in abelian groups

Lemma (Limonchenko & S. '22)

There is the following commutative diagram of group homomorphisms ($I \subseteq J \in \operatorname{cat} K$, $d = 1, 2$)

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1 & \to & G_d^I/(G_d^I \cap H_d) & \to & G_d^m/H_d & \to & G_d^m/(H_d \cdot G_d^I) & \to & 1 \\
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Definition (Stabilizer-quotient SES)

Define the (complex or real, resp.) toric (i.e. objects are tori and arrows are group morphisms) ($\operatorname{cat} K$)-diagrams by (1) as follows

$$\kappa(1) \to S_d \to \kappa(L_d) \to Q_d \to \kappa(1).$$

We call S and Q the stabilizer and the quotient diagram, respectively.

Toric configurations and diagrams

For any closed freely acting subgroup H, we obtain Yuzvinsky's toric configuration in L and Davis-Januszkiewicz space in BL as the colimits of these diagrams:

$$\operatorname{colim} S_2 = (S^1, *)^K, \ \operatorname{colim} BS_2 = (\mathbb{C}P^\infty, *)^K = DJ_K.$$

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The respective cohomology rings are exterior and symmetric Stanley-Reisner rings:

$$H^*((S^1,*)^K) \cong \mathbb{Z}\langle K\rangle = \Lambda[u_1,\dots,u_m]/(u_I|\ I \notin K),\ \deg u_i = 1,$$

$$H^*((\mathbb{C}P^\infty,*)^K) \cong \mathbb{Z}[K] = \mathbb{Z}[v_1,\dots,v_m]/(v_I|\ I \notin K),\ \deg v_i = 2.$$

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Let H be a subtorus acting freely on $\mathcal{Z}_K$. Dependence condition: Suppose that the stabilizers $T^{i-1,i+1}$, $T^{i,i+1}$ span a rank 2 subgroup in $L = T^m/H$, $i = 1, \dots, m$ ($2 \leqslant r \leqslant m$, $r :=$ codim H, cyclic order).

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Theorem (Lü & Zhang '22)

Suppose that the dependent condition holds for H. Then the cohomology groups of $\mathcal{Z}_{P^2}$, $X_{P^2,H}$ are torsion-free, and

$$\dim H^p(X_{P^2,H};\mathbb{Z}) = \binom{r-2}{p} + (m-2)\binom{r-2}{p-2} - \binom{r-2}{p-4}.$$

Part 1: Homotopy decomposition for any MAC quotient and its Borel space

Homotopy colimits in *Top*

Consider the Quillen model structure on the category $\mathcal{C} = \mathrm{Top}$ of weakly Hausdorff compactly generated topological spaces (weak equivalences/Serre fibrations/left lifting property w.r.t. acyclic fibrations).

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Fix the Reedy model structure on $\mathrm{Top}^{\mathrm{cat}\,K}$ (object-wise weak equivalences $D \to E$/object-wise Serre fibrations/$L_I E \sqcup D(I) \to E(I)$ is a cofibration for any $I \in \mathrm{cat}\,K$, where $L_I(E) := \operatorname*{colim}_{\mathrm{cat}\,K_{<I}} E$ is the latching object).

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Definition

A homotopy colimit

$$\underset{\mathrm{cat}\,K}{\mathrm{hocolim}}\, D = \mathbb{L}\underset{\mathrm{cat}\,K}{\mathrm{colim}}\, D.$$

of $D \in \mathrm{Top}^{\mathrm{cat}\,K}$ is the *left derived functor* of colim in the *Reedy model structure* $\mathrm{Top}^{\mathrm{cat}\,K}$.

Homotopy colimits in *Top*

Consider the Quillen model structure on the category $\mathcal{C} = \mathrm{Top}$ of weakly Hausdorff compactly generated topological spaces (weak equivalences/Serre fibrations/left lifting property w.r.t. acyclic fibrations).
Fix the Reedy model structure on $\mathrm{Top}^{\mathrm{cat}\,K}$ (object-wise weak equivalences $D \to E$/object-wise Serre fibrations/$L_I E \sqcup D(I) \to E(I)$ is a cofibration for any $I \in \mathrm{cat}\,K$, where $L_I(E) := \operatorname*{colim}_{\mathrm{cat}\,K_{<I}} E$ is the latching object).

Definition

A homotopy colimit

$$\operatorname*{hocolim}_{\mathrm{cat}\,K} D = \mathbb{L}\operatorname*{colim}_{\mathrm{cat}\,K} D.$$

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$$\text{Bousfield-Kan:}\ \operatorname*{hocolim}_{\mathrm{cat}\,K} D = \int^{I \in \mathrm{cat}\,K} B(I \downarrow \mathrm{cat}\,K)^{op} \times D(I).$$

Simplicial sets

Let Δ be the category consisting of nonempty finite linearly ordered sets (objects) and order-preserving maps (arrows). The arrows

$$d_{n,k}\colon [n] \to [n-1],\ s_{n,k}\colon [n-1] \to [n],$$

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A (co)simplicial object $X_\bullet \in s\mathcal{C}$ (resp. $X^\bullet \in c\mathcal{C}$) in a category $\mathcal{C}$ is a (covariant) functor $\Delta^{op} \to \mathcal{C}$ (resp. $\Delta \to \mathcal{C}$). Equivalently, $X_\bullet$ consists of simplices X_n on $[n]$ and face and degeneracy maps satisfying the (co)simplicial identities.

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The simplicial replacement srep D is a simplicial object in $\mathcal{C}$ given by arrows (easy) and objects

$$(\operatorname{srep} D)_n = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} D(i_n).$$

Cosimplicial replacement crep D is defined similarly (using products and $D(i_0)$).

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Geometric realization of $X_\bullet \in s\,\mathrm{Top}$ is the topological space given by

$$|X_\bullet| := \operatorname{coeq}\left[\bigsqcup_{[n]\to[k]} X_k\times\Delta^n \rightrightarrows \bigsqcup_n X_n\times\Delta^n\right]$$

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Main tools for hocolim

- *Projection lemma*: if D is cofibrant, then $\operatorname{hocolim} D \simeq \operatorname{colim} D$.
- *Homotopy lemma*: if a morphism $D \to E$ consists of homotopy equivalences, then $\operatorname{hocolim} D \to \operatorname{hocolim} E$ is a homotopy equivalence.

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Example

The diagrams S_d, BS_d are cofibrant: $L_I D \to D(I)$ is a closed embedding of the subset in $(S^1)^n$ or $(\mathbb{C}P^\infty)^n$, respectively.

Homotopy decompositions for MAC quotient...

Theorem (Limonchenko & S. '22)

For any closed subgroup $H_d \subseteq G_d^m$ and any K one has L_d-equivariant homotopy equivalence for the MAC quotient and diagram Q_d

$$(D^d, S^{d-1})^K / H_d \simeq \operatorname{hocolim}\ G_d^m / (H_d \cdot G_d^I),\ d = 1, 2. \tag{2}$$

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Remark

- The orbits of the L_d-action on $G_d^m/(H_d \cdot G_d^I)$ are isomorphic (because the group action is transitive), but on $(D^d, S^{d-1})^I/H_d$ they are not, $I \in \operatorname{cat} K$.
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To prove the theorem, we construct the fibration between diagrams from (2) with contractible fibers, then apply the homotopy lemma.

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Example (Partial quotient X_K)

$$H^*_{L_d}(X_K;\ R_d) \cong R_d[K].$$

EMSS for complex MAC quotients

The Eilenberg-Moore spectral sequence

$$Tor^{-i,2j}_{H^*(BL)}(H^*(\operatorname{colim} BS), \mathbb{Z}) \Rightarrow H^{2j-i}(F;\mathbb{Z}),$$

of the Serre fibration

$$F \to \operatorname{colim} BS \simeq B_L \mathcal{Z}_K/H \to BL \simeq \kappa(BL), \qquad (3)$$

converges to the cohomology of the homotopy fiber F.

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Sketch of proof. By stabilizer-quotient SES, $\mathrm{hofib}[BS(I) \to BL] = Q(I)$. By Puppe's theorem,

$$\mathrm{hofib}[\mathrm{hocolim}\, D \to \mathrm{hocolim}\, \kappa(X)] = \mathrm{hocolim}\, \mathrm{hofib}[D(I) \to X].$$

Apply this to (3) and use $\mathrm{hocolim}\, Q \simeq \mathcal{Z}_K/H$.

EMSS collapse

Theorem (Buchstaber & Panov '98, '99, '02; Panov '15; Franz '21)

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Example (Nontrivial multiplicative abutment in EMSS (M. Franz))

Let $H = \mathbb{Z}/2\mathbb{Z}$ be the diagonal subgroup in T^2, $K = \partial\Delta^1$ (two points). Then $X = X_{K,H} \simeq \mathbb{R}P^3$ and EMSS page 2 is

	$\mathbb{Z}_2$		4
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	2
		$\mathbb{Z}_2$	0
-2	-1	0	

.

Since the product in Tor-algebra respects bidegrees, the square of the generator in bidegree $(-1, 2)$ vanishes. On the other hand, the corresponding element in $H^1(\mathbb{R}P^3; \mathbb{Z}_2)$ does not square to zero.

Part 2: Bousfield-Kan spectral sequence collapse for toric diagrams and classifying spaces

Cohomological BKSS for diagrams

Definition

The (cohomological) Bousfield-Kan spectral sequence for a *simplicial topological space* X_* is the (cohomological) Leray-Serre spectral sequence corresponding to the filtration on X_* by the simplicial degree. For any $(\mathrm{cat}\, K)$-diagram D one has BKSS for srep D:

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Example

BKSS of the $(\operatorname{cat} K)$-diagrams Q, S, BS for any K, H.

BKSS collapse for toric diagrams and classifying spaces

Theorem (S. '22)

For any complex toric $(\operatorname{cat} K)$*-diagram* R*, BKSS for* R *and* BR *collapse at page 2 with trivial additive abutment.*

BKSS collapse for formal diagrams

For any $D \in [\operatorname{cat} K, \mathrm{Top}]$ the diagrams $C^*(D;\ \mathbb{Z})$, $H^*(D;\ \mathbb{Z})$ in $[\operatorname{cat}^{op} K, \mathrm{Ch}_{\mathbb{Z}}] = [\operatorname{cat} K, \mathrm{Ch}_{\mathbb{Z}}^{op}]$ are given by post-composing with normalized cochain and singular cohomology functors, respectively

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A $\mathcal{C}$-diagram D in Top is called formal if there is a chain of quasiisomorphisms between the $\mathcal{C}$-diagram $C^*(D;\mathbb{Z})$ in $\mathrm{Ch}_{\mathbb{Z}}^{op}$ and $H^*(D;\mathbb{Z})$. ($\mathcal{C} = *$: usual notion of formality for a topological space.)

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Proposition (Lambrechts, Tourchine & Volić '10)

For any formal simplicial space $X_\bullet \in \mathrm{sTop}$, the spectral sequence $((E_{X_\bullet})_r^{i,j}, d_r)$ (with integral coefficients) collapses at the second page with trivial abutment.

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For any formal $(\operatorname{cat} K)$-diagram D in Top the diagram $\operatorname{srep} D$ in sTop is formal.

Simplicial tori

For any $k \geqslant 0$, the simplicial torus is the simplicial abelian group BN ($N = \mathbb{Z}^k$), $BN_0 := \{[\,]\}$,

$$BN_n := \{[a_0, \dots, a_{n-1}] |\ a_0, \dots, a_{n-1} \in N\},\ \ n > 0,$$

and faces and degenerations are given by the formulas

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E.g. $\overline{W}(T) = B\pi_1(T, 0)$. Any lattice homomorphism $f\colon N \to N'$ induces $f_*\colon BN \to BN'$. For a topological space X denote by $S(X)$ the simplicial set of singular simplices in X. Explicit functorial homotopy equivalence yields

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Classifying spaces of simplicial tori

For any $k \geqslant 0$, the classifying space of the simplicial torus BN is the simplicial set $B(BN)$ $(N = \mathbb{Z}^k)$, $B(BN)_0 := \{[\,]\}$,

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- M, F_* are functorial w.r.t. torus and lattice homomorphisms;
- $M \circ F_*$ is the desired functorial homotopy equivalence.

Formality for toric diagrams and classifying spaces

Lemma (Limonchenko & S. '22)

Then there is the following commutative diagram with q/iso rows:

$$\begin{array}{ccccccccc} H^*(T') & \xrightarrow{\kappa} & \overline{W}^*(S^1)^{\otimes r'} & \to & \mathrm{Hom}(\overline{W}_*(S^1)^{\otimes r'};\ \mathbb{Z}) & \xleftarrow{ez^*} & \overline{W}^*(T') & \to & C^*(T') \\ \downarrow f^* & & \downarrow A^* & & \downarrow \mathrm{Hom}((A^*)^T, \mathrm{Id}) & & \downarrow f^* & & \downarrow f^* \\ H^*(T) & \xrightarrow{\kappa} & \overline{W}^*(S^1)^{\otimes r} & \longrightarrow & \mathrm{Hom}(\overline{W}_*(S^1)^{\otimes r};\ \mathbb{Z}) & \xleftarrow{ez^*} & \overline{W}^*(T) & \to & C^*(T). \end{array}$$

A similar diagram exists for classifying spaces.

Formality for toric diagrams and classifying spaces

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\downarrow f^* & & \downarrow A^* & & \downarrow \mathrm{Hom}((A^*)^T, \mathrm{Id}) & & \downarrow f^* & & \downarrow f^* \\
H^*(T) & \xrightarrow{\kappa} & \overline{W}^*(S^1)^{\otimes r} & \to & \mathrm{Hom}(\overline{W}_*(S^1)^{\otimes r};\ \mathbb{Z}) & \xleftarrow{ez^*} & \overline{W}^*(T) & \to & C^*(T).
\end{array}$$

A similar diagram exists for classifying spaces.

Remark (Denham & Suciu '07)

Formality of a toric diagram does not imply formality of the respective homotopy colimit.

Part 3: Results on BKSS page 2 using sheaf cohomology

Sheaves over posets and cohomology

A cosheaf $\mathcal{F}$ over cat K is a functor from $[\mathrm{cat}^{op} K, \mathrm{Ab}]$. A poset P is topologized by base of open sets (ideals) $P_{\geqslant p}$ (Alexandrov topology).

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Let $N(\mathrm{cat}\, K)$ be the simplicial complex $\mathrm{cone}_{\varnothing} K'$ (cone over barycentric sub of K with apex $\varnothing$):

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The refinement $r\mathcal{F}$ of $\mathcal{F}$ is the sheaf (functor from $[N(\operatorname{cat} K), \mathrm{Ab}]$) defined by

$$r\mathcal{F}(I_0 \supset I_1 \supset \cdots \supset I_s) := \mathcal{F}(I_s),$$

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Cohomology $H^s(N(\operatorname{cat} K); r\mathcal{F})$ of the sheaf is defined by the cochain complex generated by $I_0 \supset \cdots \supset I_s \times x$, where $x \in \mathcal{F}(I_s)$ and

$$d(I_0 \supset I_1 \supset \cdots \supset I_s \times x) := \sum_{i=0}^{s} (-1)^i \sum_J I_0 \supset I_1 \supset \cdots \supset I_{i-1} \supset J \supset I_i \supset \cdots \supset I_s \times x + (-1)^{s+1} \sum_J I_0 \supset I_1 \supset \cdots \supset I_s \supset J \times \mathcal{F}(J \subset I_s)(x).$$

Step 3: BKSS page 2 and cohomology of sheaves

Composition of $H^*(-;\ \mathbb{Z})$ with any $(\operatorname{cat} K)$-diagram D induces the cosheaf $H^*(D) = H^*(D;\ \mathbb{Z})$ over $\operatorname{cat} K$ (i.e. a $(\operatorname{cat}^{op} K)$-diagram with values in abelian groups, rings or algebras).

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Proposition

The group $(E_D)_2^{p,q}$ is isomorphic to each of the following:

- *Right derived functor* $\lim^p \tilde{H}^q(D)$ *of limit applied to the cohomology of the diagram;*
- *Cohomology* $H^p(\operatorname{cat} K; H^q(D)) := H^p(\operatorname{srep} \tilde{H}^q(D)^\bullet)$ *with coefficients in a functor;*
- *Cohomology* $H^p(N(\operatorname{cat} K); rH^q(D))$ *of the sheaf over poset.*

BKSS page 2 for MAC quotients

- BKSS collapse implies that the singular and equivariant cohomology of $(D^d, S^{d-1})^K/H_d$ are isomorphic to direct sums of sheaf cohomology over $N(\mathrm{cat}\, K)$.

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- For any diagram of real tori (finite $\mathbb{F}_2$-linear spaces) the page 1 of this BKSS is concentrated at row 0, by looking at the stalks of cohomology for discrete sets. So the BKSS collapse for real toric diagrams follows trivially.

Comparison SS for BKSS and EMSS

Theorem (S. '22)

There exists the spectral sequence

$$({}'E_{r,t}^{i,j}, d_r'),\ d_r' \colon {}'E_{r,t}^{i,j} \to {}' E_{r,t}^{i+r,j-r+1}, \tag{4}$$

$$ {}'E_{2,t}^{i,j} = \mathrm{Tor}_{H^*(BL)}^{-i,2t}(\lim{}^j H^*(BS), \mathbb{Z}) \Rightarrow \lim{}^{-i+j+t} \widetilde{H}^t(Q).$$

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Remark

The SS (4) measures commutativity of Tor and lim, because

$$\lim{}^j \mathrm{Tor}_{H^*(BL)}^{-i,2t}(H^*(BS), \mathbb{Z}) = \begin{cases} \lim{}^j \widetilde{H}^t(Q), & i = t; \\ 0, & \text{else.} \end{cases}$$

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Example (MAC): $\mathrm{Tor}_{H^*(BT^m)}^{-i,2t}(\mathbb{Z}[K]; \mathbb{Z}) \cong H^{-i,2t}(\mathcal{Z}_K) \cong \lim{}^{-i+t} \widetilde{H}^t(Q)$ $(j = 0)$.

SS construction outline

1. Denote by P^i the $(-i)$-th group of the Koszul resolution ($r = \operatorname{rk} L$)

$$0 \to \Lambda^r(L) \otimes H^*(BL) \to \Lambda^{r-1}(L) \otimes H^*(BL) \to \cdots \to H^*(BL) \to \mathbb{Z} \to 0.$$

2. Define the bicomplex $(C_t^{i,j}, d_h, d_v)$ by

$$C_t^{i,j} := C^j(\operatorname{cat} K;\ (P^* \otimes H^*(BS))^i_{2t}),$$

where $i = 2t - a$, so the $(-a, 2t)$-th graded component is taken, d_h is the differential for the cochain cohomology (increase j by 1) and d_v is induced by the morphism from the Koszul resolution (increase i by 1) ($H^*(BS)$ is a diagram of $H^*(BL)$-modules).

3. $$H_v^i(H_h^j(C_t^{*,*})) = \operatorname{Tor}_{H^*(BL)}^{-(2t-i),2t}(\lim{}^j H^*(BS), \mathbb{Z}),$$

4. $$H_h^j(H_v^i(C_t^{*,*})) = \begin{cases} \lim{}^j \widetilde{H}^t(Q), & i = t, \\ 0, & i \neq t. \end{cases}$$

Equivariant formality

Lemma (Franz & Puppe '07)

TFAE for a T-space X (cohomological equivariant formality over $\mathbb{Z}$):

1. *the cohomological Serre SS (over $\mathbb{Z}$) for the respective Borel construction collapses at page 2;*
2. $\iota^*\colon H_T^*(X) \to H^*(X)$ *is epi for the fiber inclusion* $\iota : X \to B_T X$;
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Freeness of the $H^*(BT)$-module $H_T^*(X;\ \mathbb{Z})$ is strictly stronger than EF over $\mathbb{Z}$ (Franz, Puppe '06). EF over $\mathbb{Q}$ is equivalent to freeness of the $H^*(BT)$-module $H_T^*(X;\ \mathbb{Q})$.

Vanishing for equivariantly formal partial quotients

For an EFPQ $X_{K,H}$ one has $H^{odd}(X_{K,H}) = 0$ (because $H_L^*(X_{K,H}) \cong \mathbb{Z}[K]$ has no odd cohomology).

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Corollary (BKSS vanishing except diagonal)

For any equivariantly formal partial quotient one has

$$\lim{}^i \widetilde{H}^j(Q) \cong \begin{cases} H^{2i}(X_K) \cong \mathrm{Tor}^{0,2i}_{H^*(BL)}(\mathbb{Z}[K];\mathbb{Z}), & i = j; \\ 0, & \textit{else}. \end{cases}$$

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Remark

Danilov '78 used different approach to prove vanishing except diagonal for sheaf cohomology over posets ($\mathbb{C}$ coefficients) in the case of a toric non-singular projective variety using Hodge structures.

Betti numbers for equivariantly formal partial quotients

Theorem (S. '23)

$$\dim H^{2i}(X_{K,H};\ \mathbb{Q}) = (-1)^{i+1} \sum_{I \in \operatorname{cat} K} \binom{r - |I|}{i} \widetilde{\chi}(\operatorname{lk} I), \tag{5}$$

$$\operatorname{lk}_K I := \{J \in K\colon\ J \sqcup I \in K\}, \quad r := \operatorname{codim} H,\ \widetilde{\chi}(X) := \sum_i (-1)^i \dim \widetilde{H}^i(X).$$

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Example (PL-manifold)

Let K be a combinatorial $(n-1)$-dimensional manifold (i.e. $\operatorname{lk}_K I \cong S^{n-\dim I}$ for all $I \in K$) and suppose that $X_{K,H}$ is an EFPQ. Then the RHS of (5) coincides with (for a toric manifold, $K = (\partial P^n)^*$ and $r = n$ this coincides with $h_i(K)$)

$$\sum_j (-1)^{n-j-i-1} \binom{r-(j+1)}{i} f_j(K),\ f_j(K) := \sharp\{I \in K|\ \dim I = j\}.$$

Subgroup prelattices

A poset is called a prelattice if any its bounded above subset has sup.
The distributive prelattice $\{T^I\}_{I\in \text{cat}\, K}$ maps to $\{T^I/(H\cap T^I)\}_{I\in \text{cat}\, K}$ with "kernel" $\{H\cap T^I\}_{I\in \text{cat}\, K}$ under the projection $T^m \to T^m/H$.

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1. $T^{I\cup J}\cap H = (T^I\cap H)\cdot(T^J\cap H)$ *if* $I, J \in K$;
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1. *The prelattice* $\{T^I/(H\cap T^I)|\ I\in K\}$ *is distributive and* $(*)$ *holds;*
2. $H^*(BS)$ *is flasque.*
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Notice that $T^{I\cup J}/(H\cap T^{I\cup J}) = T^I/(H\cap T^I)\cdot(T^J/(H\cap T^J))$ holds for all $I,J\in K$. Flasque-ness is a stronger condition than $(*)$ (fails for partial quotients).

Filtered distributivity

A sheaf $\mathcal{F}$ is flasque if $H^0(P;\mathcal{F}) \to H^0(U;\mathcal{F})$ is epi for any open $U \subseteq P$. Flasque sheaves on posets are applied to cohomology of configurations, local cohomology etc. (i.e. by S. Yuzvinsky).

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The collections $\{T^I \cap H\}_{I\in \mathrm{cat}\, K}$ and $\{T^I/(T^I \cap H)\}_{I\in \mathrm{cat}\, K}$ are the associated graded components of the collection $\{T^I\}_{I\in \mathrm{cat}\, K}$ with respect to the filtration $\{1\} \subseteq H \subseteq T^m$.

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Proposition (TFAE)

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2. $\{H\cap T^I\}_{I\in \mathrm{cat}\, K}$ *and* $\{T^I/(H\cap T^I)\}_{I\in \mathrm{cat}\, K}$ *are distributive prelattices &* $(*)$;
3. *The following identities hold*

$$T^{I\cap J} \cdot (T^I \cap H) = T^I \cdot (T^J \cap H),$$

$$T^J \cap (H \cdot T^I) = (T^J \cap H) \cdot T^{I\cap J},$$

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Filtered distributivity is stronger than flasque-ness.

Commutation of H^* and (co)limits

Theorem (Limonchenko, S. '22)

$(*) \Rightarrow rH^*(BS)$ is acyclic.

$$\tilde{H}^*_{L_d}((D^d, S^{d-1})^K/H_d;\ \mathbb{Z}) \cong \lim\ \tilde{H}^*(BS_d;\ \mathbb{Z}).$$

In particular, $\tilde{H}^{odd}_{L_2}((D^2, S^1)^K/H_2;\ \mathbb{Z}) = 0$ holds and $\operatorname{colim} BS$ is formal.

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Idea of pf: Notbohm & Ray '07 (construct a twin diagram to BS_d). Under stronger condition $(**)$ the proof is elementary: flasque $\Rightarrow$ acyclic.

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Corollary (Limonchenko & S. '22)

$(*) \Rightarrow$ the Eilenberg-Moore SS

$$\operatorname{Tor}^{-i,2j}_{H^*(BL)}(H^*(\operatorname{colim} BS);\mathbb{Z}) \Rightarrow H^{2j-i}(\mathcal{Z}_K/H),$$

of the Borel construction collapses at page 2.

Torsion in cohomology of partial quotients

Let K be such that $\mathrm{lk}_{K_I} J$ has torsion-free cohomology for any $I, J \subseteq [m]$.

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Lemma

Let $\mathcal{F}$ be a torsion-free cosheaf with all arrows monomorphic and having torsion-free cokernels. Under the above assumption on K, $H^i(N(\mathrm{cat}\, K);\ \mathcal{F})$ is torsion-free and

$$H^i(N(\mathrm{cat}\, K);\ r\mathcal{F}) \to H^i(N(\mathrm{cat}\,\mathrm{lk}\, v);\ r\mathcal{F}),$$

has torsion-free cokernel for any $v \in [m]$.

Sketch of proof (Lemma)

Prove that cohomology groups are torsion-free. 1) $K = \Delta_m$ (acyclic case); 2) star $w \neq K$ for some $w \in [m]$. Induction on the number of simplices in K.

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$$\cdots \to H^i(N(\operatorname{cat} K);\ r\mathcal{F}) \to$$

$$H^i(N(\operatorname{cat}\operatorname{star} v);\ r\mathcal{F}|_{\operatorname{star} v}) \oplus H^i(N(\operatorname{cat} K_{[m]\setminus v});\ r\mathcal{F}|_{K_{[m]\setminus v}}) \xrightarrow{f}$$

$$H^i(N(\operatorname{cat}\operatorname{lk} v);\ r\mathcal{F}|_{\operatorname{lk} v}) \to H^{i+1}(N(\operatorname{cat} K);\ r\mathcal{F}) \to \cdots .$$

Sketch of proof (Lemma)

Prove that cohomology groups are torsion-free. 1) $K = \Delta_m$ (acyclic case); 2) $\operatorname{star} w \neq K$ for some $w \in [m]$. Induction on the number of simplices in K. Sheaf Mayer-Vietoris LES for $K = \operatorname{star} v \cup_{\operatorname{lk} v} K_{[m]\setminus v}$:

$$\cdots \to H^i(N(\operatorname{cat} K);\ r\mathcal{F}) \to$$

$$H^i(N(\operatorname{cat}\operatorname{star} v);\ r\mathcal{F}|_{\operatorname{star} v}) \oplus H^i(N(\operatorname{cat} K_{[m]\setminus v});\ r\mathcal{F}|_{K_{[m]\setminus v}}) \xrightarrow{f}$$

$$H^i(N(\operatorname{cat}\operatorname{lk} v);\ r\mathcal{F}|_{\operatorname{lk} v}) \to H^{i+1}(N(\operatorname{cat} K);\ r\mathcal{F}) \to \cdots .$$

Simplicial complexes $\operatorname{star} v$, $K_{[m]\setminus v}$, $\operatorname{lk} v$ satisfy the condition on link restrictions. Then the groups (except maybe for $\operatorname{cat} K$) are torsion-free and $\operatorname{coker} f$ is torsion-free by the induction. Hence, $H^i(N(\operatorname{cat} K);\ r\mathcal{F})$ is torsion-free by exactness.

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Long exact sequence of $(N(\operatorname{cat} K), N(\operatorname{lk} v))$ is obtained from

$$1 \to j_! j^{-1} r\mathcal{F} \to r\mathcal{F} \to i_* i^{-1} r\mathcal{F} \to 1,$$

The sheaves satisfy the induction assumption, so the cokernel is torsion-free.

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どうもありがとうございます。

Thank you for your attention!

Identities for link restrictions

$$(\mathrm{lk}_K I)_J = \mathrm{lk}_{K_J} I,$$

$$(\mathrm{lk}_{K_{[m]\setminus\{v\}}} I)_J = (\mathrm{lk}_K I)_{([m]\setminus\{v\})\cap J},$$

$$(\mathrm{lk}_{\mathrm{star}\, v} I)_J = \begin{cases} \mathrm{lk}_K I, \text{ if } v \in I, \\ \mathrm{cone}_v(\mathrm{lk}_K I \cap \mathrm{lk}_K v), \text{ otherwise.} \end{cases}$$

$$(\mathrm{lk}_{\mathrm{lk}_K v} I)_J = \mathrm{lk}_K I \sqcup v.$$

Functors for sheaves

$$i\colon N(\mathrm{cat}\,\mathrm{lk}\, I) \to N(\mathrm{cat}\, K),$$

$$j\colon N(\mathrm{cat}\, K) \setminus N(\mathrm{cat}\,\mathrm{lk}\, I) \to N(\mathrm{cat}\, K).$$

Sheaves $\mathcal{F}$, $\mathcal{G}$ on posets P, Q, resp. Morphism $f\colon P \to Q$.

$$\text{Pushforward}\colon f_*\mathcal{F}(Q_{\geqslant y}) := \lim_{f(x)\geqslant y} \mathcal{F}(P_{\geqslant x}).$$

$$\text{Pullback}\colon f^*\mathcal{G}(P_{\geqslant x}) := \mathcal{G}(Q_{\geqslant f(x)}).$$

$$\text{Pushforward w/open supp}\colon f_!\mathcal{F}(Q_{\geqslant y}) := \operatorname*{colim}_{f(x)\leqslant y} \mathcal{F}(P_{\geqslant x}).$$