Equivariant K-theory of Springer Varieties

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Springer Variety of type A

- Let $\mathbb{C}^n$ denote the n-dimensional complex vector space.
- Let $\mathcal{F} = \mathcal{F}(\mathbb{C}^n)$ denote

  $$\{\underline{V} := (0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim(V_i) = i \; \forall \; i\}$$

  the variety of full flags in $\mathbb{C}^n$.
- Fix $N : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a nilpotent linear transformation of $\mathbb{C}^n$
- Let $\mathcal{F}_N$ denote the following subvariety of $\mathcal{F}$

  $$\{\underline{V} \in \mathcal{F} \mid NV_i \subseteq V_{i-1} \; \forall \; 1 \leq i \leq n\}$$

  called the Springer variety of type A associated to N.

Notations

- Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ be the sizes of the nilpotent Jordan blocks of the Jordan canonical form of N.
- Then $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a partition of n associated to N.
- For $g \in GL_n(\mathbb{C})$, $\mathcal{F}_N$ is isomorphic to $\mathcal{F}_{gNg^{-1}}$ as algebraic varieties.
- Thus w.l.g we shall assume N itself is in the Jordan Canonical form.
- When $\lambda = (1, 1, \ldots, 1)$ then $N = 0$ and $\mathcal{F}_{(1,1,\ldots,1)} = \mathcal{F}(\mathbb{C}^n)$.
- On the other hand when $\lambda = (n)$ then N is regular nilpotent so that $\mathcal{F}_{(n)}$ is a one point variety consisting of the standard flag

$$\{\underline{E} : (0) = E_0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{n-1} \rangle \subset \mathbb{C}^n\}$$

Further Notations

- Let $T^n \simeq (S^1)^n$ denote the n-dimensional torus the

$$T^n = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \;\middle|\; t_i \in \mathbb{C}\,,\ |t_i| = 1\,, 1 \leq i \leq n \right\}$$

- T^n acts on $\mathcal{F}(\mathbb{C}^n)$ but not on $\mathcal{F}_\lambda$ in general.
- We therefore consider the l-dimensional subtorus of T^n which acts on $\mathcal{F}_\lambda$.

$$T^l = \left\{ \begin{pmatrix} h_1 I_{\lambda_1} & & & \\ & h_2 I_{\lambda_2} & & \\ & & \ddots & \\ & & & h_l I_{\lambda_l} \end{pmatrix} \;\middle|\; h_i \in \mathbb{C}\,,\ |h_i| = 1\,, 1 \leq i \leq l \right\}$$

- where I_{λ_i} denotes the identity matrix of size λ_i for $1 \leq i \leq l$.

Background and History

- The Springer variety $\mathcal{F}_\lambda$ as the name suggests was first studied by T.A.Springer [10], [11] Hotta-Springer [5] (1976, 1978, 1977).
  - There exists an S_n action on $H^*(\mathcal{F}_\lambda;\mathbb{C})$ compatible with standard action on $H^*(\mathcal{F};\mathbb{C})$.
  - $H^*(\mathcal{F}_\lambda;\mathbb{C})$ is isomorphic as an S_n-representation to $M_\lambda := \mathrm{Ind}_{S_{\lambda_1}\times\cdots\times S_{\lambda_l}}^{S_n}(1_{S_{\lambda_1}\times\cdots\times S_{\lambda_l}})$.
- Spaltenstein [9] (1976) gave an algebraic cellular decomposition for $\mathcal{F}_\lambda$ with $\binom{n}{\lambda} := \frac{n!}{\lambda_1!\cdots\lambda_l!}$ cells where $(\mathcal{F}_\lambda)^{T^l} = \binom{n}{\lambda}$.

Background and History

- De Concini and Procesi [3] (1981) described $H^*(\mathcal{F}_\lambda;\mathbb{C})$ as the coordinate ring of a non-reduced scheme $\mathbb{C}[\mathfrak{t}_\mathbb{C} \cap \overline{O_{\lambda^\vee}}]$ the intersection of the Lie algebra $\mathfrak{t}$ of T^n with the closure of orbit of $N = J_{\lambda^\vee}$ in $M_n(\mathbb{C})$.

- Tanisaki [13] (1982) described $H^*(\mathcal{F}_\lambda;\mathbb{C})$ as quotient of polynomial ring in n variables over $\mathbb{C}$ by an ideal called the Tanisaki ideal.

- Tanisaki's presentation holds for the ring $H^*(\mathcal{F}_\lambda;\mathbb{Z})$.

Recent work

- Abe and Horiguchi [1](2016) gave a description of T^l-equivariant cohomology ring of $\mathcal{F}_\lambda$ with integer coefficients.
- $H^*_{T^l}(\mathcal{F}_\lambda;\mathbb{Z}) := H^*(ET^l \times_{T^l} \mathcal{F}_\lambda;\mathbb{Z})$ where $ET^l \longrightarrow BT^l$ is the universal principal T^l-bundle; BT^l—classifying space.
- Abe and Horiguchi [1] gave a presentation of $H^*_{T^l}(\mathcal{F}_\lambda;\mathbb{Z})$ as a quotient of a polynomial ring over $H^*(BT^l;\mathbb{Z})$ modulo the equivariant analogue of the Tanisaki ideal.

Notations

- Consider the inclusion $\iota : \mathcal{F}_\lambda \hookrightarrow \mathcal{F}$.
- Let $\mathcal{L}_i$ for $1 \leq i \leq n$ denote the tautological line bundles on $\mathcal{F}$ whose fibre over a flag $\underline{V} = (0) = V_0 \subset \cdots \subset V_n = \mathbb{C}^n$ is V_i/V_{i-1}.
- $\mathcal{L}_i$ is T^n-equivariant line bundle w.r.t standard T^n-action on $\mathcal{F}$.
- $L_i := \mathcal{L}_i \mid_{\mathcal{F}_\lambda}$ the restriction of $\mathcal{L}_i$ to $\mathcal{F}_\lambda$ for $1 \leq i \leq n$ is a T^l-equivariant line bundle on $\mathcal{F}_\lambda$.
- $c_1^{T^n}(\mathcal{L}_i) \in H^*_{T^n}(\mathcal{F})$ (respectively $c_1^{T^l}(L_i) \in H^*_{T^l}(\mathcal{F}_\lambda)$) denote the T^n-equivariant (respectively the T^l-equivariant) Chern class of $\mathcal{L}_i$ (respectively L_i) for $1 \leq i \leq n$.

Further notations

- Let $\lambda^{\vee}$ denote the partition of n dual to λ.
- Let $p_{\lambda^{\vee}}(s) = \lambda^{\vee}_{n-s+1} + \cdots + \lambda^{\vee}_{n}$ for $1 \leq s \leq n$.
- For example when $n = 8$, $\lambda = (3, 2, 2, 1)$ then $\lambda^{\vee} = (4, 3, 1)$. Here $l = 3$. Thus $p_{\lambda^{\vee}}(s) = 0$ for $1 \leq s \leq 5$ and $p_{\lambda^{\vee}}(6) = 1$, $p_{\lambda^{\vee}}(7) = 4$, $p_{\lambda^{\vee}}(8) = 8$.

Further Notations

Let $\phi_\lambda : [n] \longrightarrow [l]$ is defined by

$$(u_{\phi_\lambda(1)}, \ldots, u_{\phi_\lambda(n)}) =$$
$$(\underbrace{u_1, \ldots, u_1}_{\lambda_1 - \lambda_2}, \underbrace{u_1, u_2, \ldots, u_1, u_2}_{2(\lambda_2 - \lambda_3)}, \ldots, \underbrace{u_1, \ldots, u_l, \ldots, u_1, \ldots, u_l}_{l(\lambda_l - \lambda_{l+1})})$$

as ordered sequences where

- $\forall \ 1 \leq r \leq l$
- the rth sector of the right hand side consists of $(u_1, u_2, \ldots, u_r)$ repeated $(\lambda_r - \lambda_{r+1})$-times
- $\lambda_{l+1} = 0$.

T^l-equivariant analogue of Tanisaki's ideal

Definition: (Abe and Horiguchi [1])

- Let $\mathcal{S} := H^*(BT^l)[y_1, \ldots, y_n]$ be the polynomial ring in n variables. Here $H^*(BT^l) = \mathbb{Z}[u_1, \ldots, u_l]$ where u_i denotes $c_1(ET^l \times_{T^l} \mathbb{C}_i)$, $1 \leq i \leq l$ associated to the character of T^l correponding to the ith coordinate projection of T^l.
- Let $\mathcal{J}_\lambda \subseteq \mathcal{S}$ be the ideal generated by the following elements

$$\sum_{k=0}^{d} e_k(y_{i_1}, \ldots y_{i_s}) \cdot h_{d-k}(u_{\phi_\lambda(1)}, \ldots, u_{\phi_\lambda(s+1-d)})$$

where

- e_k stands for the kth elementary symmetric function
- h_{d-k} stands for the $(d-k)$th complete symmetric function,
- $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, $1 \leq s \leq n$
- $q := p_{\lambda^\vee}(s)$
- $d \geq s + 1 - q$.

Theorem:(Abe and Horiguchi)[1]

- Let

$$\Phi_\lambda : \mathcal{S} \longrightarrow H^*_{T^l}(\mathcal{F}_\lambda)$$

be defined by $\Phi_\lambda(y_j) = c_1^{T^l}(L_j)$ for $1 \leq j \leq n$.

- Then Φ_λ is surjective ring homorphism.

- The kernel $\mathrm{Ker}(\Phi_\lambda) = \mathcal{J}_\lambda$ where $\mathcal{J}_\lambda$ is the ideal defined above.

Topological equivariant K-theory [8]

- X–G-space ; G– compact connected Lie group.
- G-equivariant$K$- group $K_G^0(X)$— a free abelian group on the isomorphism classes $[V]$ of G-equivariant complex vector bundles V on X modulo the relation $[V]-[V']-[V'']$ whenever $V \simeq V' \oplus V''$.
- $K_G^0(X)$ has a ring structure $[V]\cdot[V'] := [V \otimes V']$ under the operation of tensor product of G-equivariant vector bundles.
- Trivial vector bundle I_X of rank 1 with trivial G-action on the fibre is the identity element.
- $K_G^0(pt) = R(G)$— ring of finite dimensional complex representations of G.
- The map $X \longrightarrow pt$ induces an $R(G)$-algebra structure on $K_G^0(X)$ by pull back of G-equivariant vector bundles.

T^l-equivariant K-theoretic Tanisaki ideal

Definition: (———, 2023 [15])

- Let $\mathcal{R} := R(T^l)[x_1, \dots x_n]$ where $R(T^l) = K^0_{T^l}(pt) = \mathbb{Z}[u_i^{\pm 1} : 1 \leq i \leq l]$ where u_i is the character of T^l corresponding to the ith coordinate projection for $1 \leq i \leq l$.
- Let $\mathcal{I}_\lambda$ denote the ideal in $\mathcal{R}$ generated by the elements:
  - $$\sum_{0 \leq k \leq d} (-1)^{d-k} e_k(x_{i_1}, \dots, x_{i_s}) \cdot h_{d-k}(u_{\phi_\lambda(1)}, \dots, u_{\phi_\lambda(s+1-d)})$$
    .
  - $1 \leq s \leq n$
  - $1 \leq i_1 < \cdots < i_s \leq n$
  - $q := p_{\lambda^\vee}(s)$
  - $d \geq s + 1 - q$

Main Theorem:(———-, 2023[15])

- With the above notations, let
- $\Psi_\lambda : \mathcal{R} \longrightarrow K^0_{T^l}(\mathcal{F}_\lambda)$ be the ring homomorphism defined by
- $\Psi_\lambda(x_j) = [L_j]_{T^l}$ for $1 \leq j \leq n$.
- The homomorphism Ψ_λ is surjective and
- $ker(\Psi_\lambda) = \mathcal{I}_\lambda$.

Cellular structute of $\mathcal{F}_\lambda$ and its consequences

- Recall that (see Spaltenstein, 1976 [9]), $\mathcal{F}_\lambda$ has an algebraic cellular decomposition with locally closed cells isomorphic to affine spaces with $\binom{n}{\lambda} := \frac{n}{\lambda_1!\cdots\lambda_l!}$ cells which is also equal to the number of T^l-fixed points in $\mathcal{F}_\lambda$ (see [1]).
- This cellular decomposition is T^l-stable. Indeed the cells arise as intersections of $\mathcal{F}_\lambda$ with the T^n-invariant Schubert cells in $\mathcal{F}$ (see J. Tymoczko, 2016[14]).
- We have the following result proved more generally for any T-equivariant K-group of T-cellular varieties:
- **Theorem 1:** [15, Theorem 5.1] The ring $K^0_{T^l}(\mathcal{F}_\lambda)$ is a free $R(T^l)$-module of rank $\binom{n}{\lambda}$.

- The inclusion $\iota_\lambda : \mathcal{F}_\lambda \hookrightarrow \mathcal{F}$ induces the pull-back map

$$\iota_\lambda^! : K^0_{T^n}(\mathcal{F}) \longrightarrow K^0_{T^l}(\mathcal{F}_\lambda).$$

  by pull back of equivariant vector bundles.
- $K^0_{T^l}(\mathcal{F}_\lambda)$ gets an $R(T^n)$-module structure via the map $R(T^n) \longrightarrow R(T^l)$ induced by the inclusion $T^l \hookrightarrow T^n$.
- Using the T^l-invariant algebraic cellular decomposition of $\mathcal{F}_\lambda$ it follows that:
- **Theorem 2:**[15, Theorem 5.3] The map $\iota_\lambda^!$ is a surjective homomorphism of $R(T^n)$-algebras.

The presentation for $K_{T^n}(\mathcal{F})$

- From the well known description of $K_{T^n}(\mathcal{F})$ due to McLeod [7] and Kostant and Kumar [6] we get the following presentation.
- **Theorem 3:** The map which sends x_i to $[\mathcal{L}_i]$ for $1 \leq i \leq n$ induces the following isomorphism of $R(T^n)$-algebras:

$$\frac{R(T^n)[x_i, \ldots, x_n]}{\langle e_k(x_1, \ldots, x_n) - e_k(t_1, \ldots, t_n) \mid 1 \leq k \leq n \rangle} \simeq K_{T^n}(\mathcal{F})$$

 where $R(T^n) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. In particular, $K_{T^n}(\mathcal{F})$ is generated by $[\mathcal{L}_i]_{T^n}$ for $1 \leq i \leq n$ as an $R(T^n)$-algebra.

- **Corollary 4:** From Theorem 2 and Theorem 3 above it follows that the $K_{T^l}(\mathcal{F}_\lambda)$ is generated by $[L_i]_{T^l}$ for $1 \leq i \leq n$ as an $R(T^l)$-algebra.

The action of the symmetric group S_n on $K_{T'}(\mathcal{F})$

- We have an S_n-action on $\mathcal{F}$ given by

$$((0) \subset \langle v_1 \rangle \subset \cdots \subset \langle v_1, \ldots, v_n \rangle) \cdot w :=$$
$$((0) \subset \langle v_{w(1)} \rangle \subset \cdots\cdots \subset \langle v_{w(1)}, \ldots, v_{w(n)} \rangle)$$

for $w \in S_n$ and $((0) \subset \langle v_1 \rangle \subset \cdots \subset \langle v_1, \ldots, v_n \rangle) \in \mathcal{F}$.

- The pull back of the line bundle $\mathcal{L}_i$ whose fibre over $V_\bullet$ is V_i/V_{i-1} under the above action is $\mathcal{L}_{w(i)}$.

- This induces a map on $K_{T^n}(\mathcal{F})$ given by $w \cdot t_i = t_i$ and $w \cdot x_i = x_{w(i)}$ by the presentation given in Theorem 3.

Localization

- The T^l-fixed points $\mathcal{F}_\lambda^{T^l}$ are indexed by the right coset representatives of $S_{\lambda_1} \times \cdots S_{\lambda_l}$ in S_n (see [1]).
- **Lemma 5:**[15, Lemma 5.4] The canonical restriction map

$$K_{T^l}(\mathcal{F}_\lambda) \xrightarrow{\iota_2} K_{T^l}((\mathcal{F}_\lambda^{T^l})) \simeq \prod_{i=1}^{m} R(T^l)$$

  is injective where $m := \binom{n}{\lambda}$.
- We have the commuting square:

$$\begin{array}{lcl} K_{T^n}(\mathcal{F}) & \xrightarrow{\iota_1} & K_{T^n}(\mathcal{F}^{T^n}) = \bigoplus_{w \in S_n} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \\ \iota_\lambda^! \downarrow & & \pi \downarrow \\ K_{T^l}(\mathcal{F}_\lambda) & \xrightarrow{\iota_2} & K_{T^l}(\mathcal{F}_\lambda^{T^l}) = \bigoplus_{w \in \mathcal{F}_\lambda^{T^l} \subseteq S_n} \mathbb{Z}[u_1^{\pm 1}, \ldots, u_l^{\pm 1}] \end{array}$$

  Here π is the canonical projection $R(T^n) \longrightarrow R(T^l)$ for $w \in \mathcal{F}_\lambda^{T^l}$ and the zero map on the other factors.

The action of the symmetric group S_n on $K_{T^l}(\mathcal{F}_\lambda)$

- We then have the following result:

- **Proposition 6:**[15, Proposition 5.5] There is an S_n action on $K^0_{T^l}(\mathcal{F}_\lambda)$ such that the map $\iota^!_\lambda : K_{T^n}(\mathcal{F}) \longrightarrow K_{T^l}(\mathcal{F}_\lambda)$ is S_n-equivariant.
- By Corollary 4 this action is defined by $w \cdot [L_i]_{T^l} := [L_{w(i)}]$ for $w \in S_n$ for $1 \leq i \leq n$.

Sectioning Canonical bundles T^l-equivariantly over $\mathcal{F}_\lambda$

- We have the following proposition:

Proposition 7: Let $1 \leq s \leq n$ and $1 \leq i_1 < \cdots < i_s \leq n$. Then

$$L_{i_1} \oplus \cdots \oplus L_{i_s} \simeq \xi \oplus \epsilon_{j_1} \oplus \cdots \oplus \epsilon_{j_q} \tag{1}$$

for some

- T^l-equivariant complex vector bundle ξ of rank $s - q$.
- T^l-equivariant trivial line bundles ϵ_{j_l} for $1 \leq l \leq q$ where
- $q := p_{\lambda^\vee}(s)$.
- For $d \geq s + 1 - q$ (1) also implies

$$L_{i_1} \oplus \cdots \oplus L_{i_s} \simeq \xi' \oplus \epsilon_{j_1} \oplus \cdots \oplus \epsilon_{j_{s+1-d}} \tag{2}$$

where $\xi' = \xi \oplus \epsilon_{s-d+2} \oplus \cdots \oplus \epsilon_q$ is a T^l-equivaraint vector bundle of rank $d - 1$.

- **Lemma: ([13, Proposition 3])**
  - $p_{\lambda^\vee}(s) = rank(N^{n-s})$ for $1 \leq s \leq n$.
  - Let $\underline{U} = (0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = \mathbb{C}^n$ be a full flag which refines the partial flag $0 = \text{Im}(N^{\lambda_1}) \subset \text{Im}(N^{\lambda_1 - 1}) \subset \cdots \subset \text{Im}(N^2) \subset \text{Im}(N) \subset \mathbb{C}^n$. Then for any $\underline{V} \in \mathcal{F}_\lambda$ and $s \geq 1$ we have $U_q \subseteq V_s$ where $q := p_{\lambda^\vee}(s)$.
- By replacing N by a conjugate gNg^{-1} we can assume w.l.g that $\text{Im}(N^{n-s}) = U_q = \mathbb{C}^q$.
- Consider the projection $\pi_s : \mathcal{F} \longrightarrow Gr_{n,s}$ sending $\underline{V} \mapsto V_s$.
- The image of the composition $\mathcal{F}_\lambda \overset{\iota_\lambda}{\hookrightarrow} \mathcal{F} \overset{\pi}{\longrightarrow} Gr_{n,s}$ is $Y_q := \{U \in Gr_{n,s} \mid U \supset U_q = \mathbb{C}^q\}$.
- Since T^l commutes with N all maps are T^l-equivariant.

Splitting of the tautological bundle over Y_q

- $\gamma_{n,s}$— tautological complex vector bundle of rank s whose fibre over $A \in Gr_{n,s}$ is the vector space A.
- $\gamma_{n,s}$ is T^l-equivariant with repect to the canonical T^l- action on $Gr_{n,s}$ which takes an s-dimensional subspace of $\mathbb{C}^n$ to another s-dimensional subspace.
- $\gamma_{n,s}\mid_{Y_q} \cong \omega \oplus (Y_q \times U_q)$ which is T^l- equivariant splitting.
- $Y_q \times U_q \cong \epsilon_{u_{\phi_\lambda(1)}} \oplus \cdots \oplus \epsilon_{u_{\phi_\lambda(q)}}$.

The λ operations in equivariant K-theory [2]

- For $x \in K_T(X)$, $\lambda_t(x) := \sum_{i \geq 0} \lambda^i(x) t^i \in K_T(X)[[t]]$.
- $\lambda^0(x) = 1$.
- For ξ a line bundle $\lambda_t(\xi) = 1 + [\xi] \cdot t$.
- $\lambda_t(x + y) = \lambda_t(x) \cdot \lambda_t(y)$.
- By (2) of Proposition 7 it follows that:

$$\lambda_t(\xi') = \prod_{1 \leq r \leq s} (1 + [L_{i_r}] \cdot t) \prod_{1 \leq l \leq s-d+1} (1 + [\epsilon_{j_l}] \cdot t)^{-1} \qquad (3)$$

for $d \geq s + 1 - d$ and $q := p_{\lambda^\vee}(s)$.

- Since $rank(\xi') = d - 1$, $\lambda_t(\xi')$ is a polynomial of degree $d - 1$ since $\lambda^k(\xi') = 0$ for $k \geq d$.
- Comparing coefficients of t^d in (3) we get

$$\sum_{0 \leq k \leq d} (-1)^{d-k} e_k([L_{i_1}], \ldots, [L_{i_s}]) \cdot h_{d-k}([\epsilon_{j_1}], \ldots, [\epsilon_{s+1-d}]) = 0 \qquad (4)$$

for $d \geq s + 1 - q$.

Proof of the main theorem

- By Corollary 4 the $R(T^l)$-algebra map $\psi_\lambda : \mathcal{R} \longrightarrow K_{T^l}(\mathcal{F}_\lambda)$ which sends x_j to $[L_j]_{T^l}$ for $1 \le j \le n$ is surjective.
- T^l acts on $\mathbb{C}^n$ via $(u_{\phi_\lambda(1)}, \ldots, u_{\phi_\lambda(n)})$.
- T^l acts on the fibre of $\epsilon_{j_1} \oplus \cdots \oplus \epsilon_{j_q}$ via $(u_{\phi_\lambda(1)}, \ldots, u_{\phi_\lambda(q)})$.
- By (4) it follows that *kernel*(ψ_λ) contains the elements

$$\sum_{0 \le k \le d} (-1)^{d-k} e_k(x_{i_1}, \ldots, x_{i_s} \cdot h_{d-k}(u_{\phi_\lambda(1)}, \ldots, u_{\phi_\lambda(q)})$$

  for $1 \le s \le n$, $1 < i_1 < \cdots < i_s \le n$ and $d \ge s + 1 - q$.
- We can show that $\mathcal{R}/\mathcal{I}_\lambda$ is generated as an $R(T^l)$-module by $\binom{n}{\lambda}$ elements.
- $\overline{\psi_\lambda} : \mathcal{R}/\mathcal{I}_\lambda \longrightarrow K_{T^l}(\mathcal{F}_\lambda)$ is a surjective $R(T^l)$-module map between two free $R(T^l)$-modules of the same rank and hence an isomorphism.

Relation with the ordinary K-ring of $\mathcal{F}_\lambda$

- By [12, Proposition 3.1] $K(\mathcal{F}_\lambda)$ is a free $\mathbb{Z}$-module of rank $\binom{n}{\lambda}$ and generated by $[L_i]$ for $1 \leq i \leq n$ as a $\mathbb{Z}$-algebra.
- The augmentation map $R(T^l)\longrightarrow\mathbb{Z}$ maps u_i to 1 for $1 \leq i \leq l$ gives $\mathbb{Z}$ as $R(T^l)$-module structure.
- We have the following:

Corollary 8: The map

$$\theta \otimes f : \mathbb{Z} \otimes_{R(T^l)} K_{T^l}(\mathcal{F}_\lambda)\longrightarrow K(\mathcal{F}_\lambda)$$

where $\theta(n)$ is the class of the trivial vector bundle of rank n and $f : K_{T^l}(\mathcal{F}_\lambda)\longrightarrow K(\mathcal{F}_\lambda)$ is the forgetful map, is an isomorphism of $\mathbb{Z}$-modules. In other words the Springer variety is weakly equivariantly formal for K-theory [4].

Presentation for the ordinary K-ring of $\mathcal{F}_\lambda$

- Putting $u_i = 1$ in the presentation of $K^*_{T^l}(\mathcal{F}_\lambda)$ given in the main theorem we recover the following presentation due to (—— and Parameswaran Sankaran, 2022) [12] for the ordinary K-ring $K^0(\mathcal{F}_\lambda)$:
- **Theorem 9:** (——and Parameswaran Sankaran, 2022 [12])$R/I_\lambda \cong K^0(\mathcal{F}_\lambda)$ where $R = \mathbb{Z}[x_1, \ldots, x_n]$ and J_λ is the K-theoretic Tanisaki's ideal generated by the elements
- $$e_d(x_{i_1}, \ldots, x_{i_s}) \cdot \binom{q+d-k-1}{q-1}$$

  where e_k is the kth elementary symmetric function and
- $1 \leq s \leq n$
- $1 \leq i_1 < i_2 < \cdots < i_s \leq n$
- $q := p_{\lambda^\vee}(s)$
- $d \geq s + 1 - q$.

THANK YOU!

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