Gale dual of GKM graph

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May 15th 2020
Workshop on Torus Actions in Topology
Fields Institute (on zoom)

§1 Gale dual (configuration)

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Definition

Gale dual (configuration) of $\mathcal{A}$ is the following configuration

$$\mathcal{B} = \{b_1, \ldots, b_m\} \subset \mathbb{R}^{m-n}$$

such that the following sequence is exact:

$$0 \longrightarrow \mathbb{R}^{m-n} \xrightarrow{B^T} \mathbb{R}^m \xrightarrow{A} \mathbb{R}^n \longrightarrow 0$$

where $B = [b_1 \cdots b_m]$ is the $(m-n) \times m$ matrix obtained by $\mathcal{B}$.

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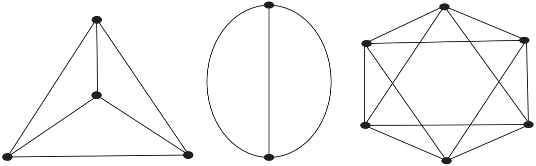
- *$\mathcal{B}$ is unique up to $GL_{m-n}(\mathbb{R})$-action and an order of vectors;*
- *Gale dual can be also defined for $\mathbb{Z}^n$ (Rossi-Teraccini).*

§2 (Abstract) GKM graph

Let $\Gamma = (V, E)$ be an *m-valent graph*, i.e., $\#E_p = m$ for all $p \in V$.



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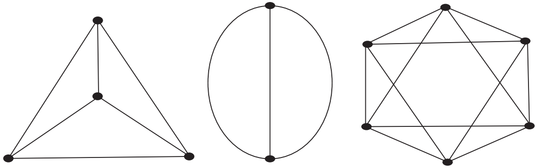

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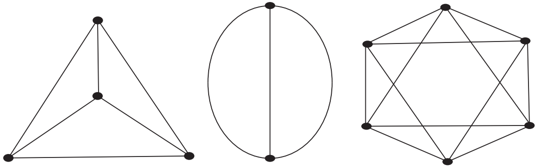

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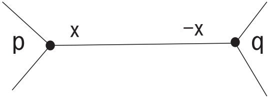
Definition

A GKM graph is a labelled graph (Γ, α, ∇) whose label $\alpha : E \to H^2(BT^n) \simeq \mathbb{Z}^n$ (for $1 \leq n \leq m$) satisfies the following conditions:

Axial function α

$\alpha : E \to H^2(BT^n) \simeq \mathbb{Z}^n$ (called axial function) satisfies the following three conditions:

(1) $\alpha(pq) = -\alpha(qp)$



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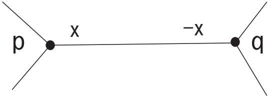

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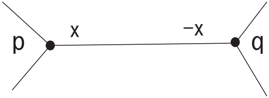
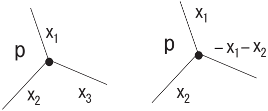


(2) $\{\alpha(e) \mid e \in E_p\}$ spans $\mathbb{Z}^n$ (effectiveness) and pairwise linearly independent



where $H^2(BT^3) = \langle x_1, x_2, x_3 \rangle$ (left) and $H^2(BT^2) = \langle x_1, x_2 \rangle$ (right).

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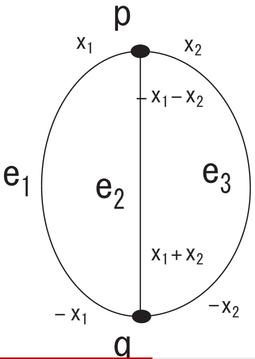
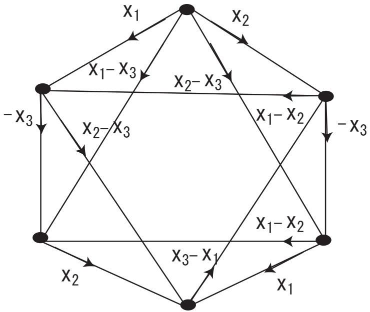
Example





§3 Gale dual of a GKM graph (Γ, α, ∇)

IDEA

The axial functions on each vertices may be regarded as a vector configuration (in $\mathbb{Z}^n$).

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Problem

Can we say something about the Gale dual of axial functions on each vertex?

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Then, the axial function defines the following (surjective) homomorphism:

$$\mathbb{Z}E_p (\simeq \mathbb{Z}^m) \xrightarrow{\alpha_p} H^2(BT^n) (\simeq \mathbb{Z}^n)$$

induced from

$$\alpha_p : e_i \mapsto \alpha(e_i), \quad \alpha_p = \begin{bmatrix} \alpha(e_1) & \cdots & \alpha(e_m) \end{bmatrix}.$$

Definition

Let ρ_p be a Gale dual of α_p, i.e.,

$$\rho_p = \{\rho_p(e_1), \ldots, \rho_p(e_m)\} \subset \mathbb{Z}^{m-n}$$

which satisfies that the following sequence is exact:

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Definition (K-Matsumura-Yukitou)

We say (Γ, ρ, ∇) a Gale dual of GKM graph (Γ, α, ∇).

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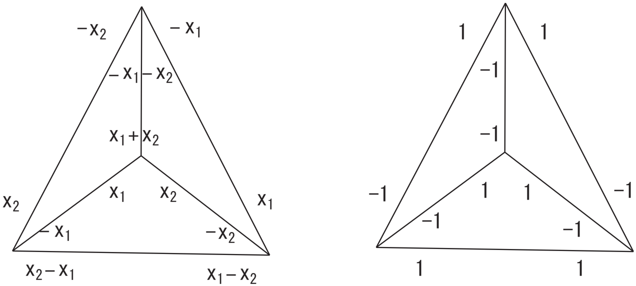

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Theorem (K-Yukitou)

Let (Γ, α, ∇) be a complexity one GKM graph and (Γ, ρ, ∇) be its Gale dual. Then, for $\{\rho(e_1), \ldots, \rho(e_{n+1})\} \subset \mathbb{Z}$ around each vertex,

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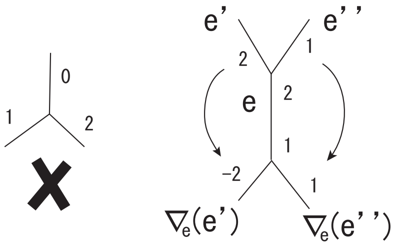

§4 Theorem 1

Assume the GKM graph (Γ, α, ∇), $\alpha : E \to H^2(BT^n)$, extends to a torus graph $(\Gamma, \widetilde{\alpha}, \nabla)$, $\widetilde{\alpha} : E \to H^2(BT^{n+1})$.

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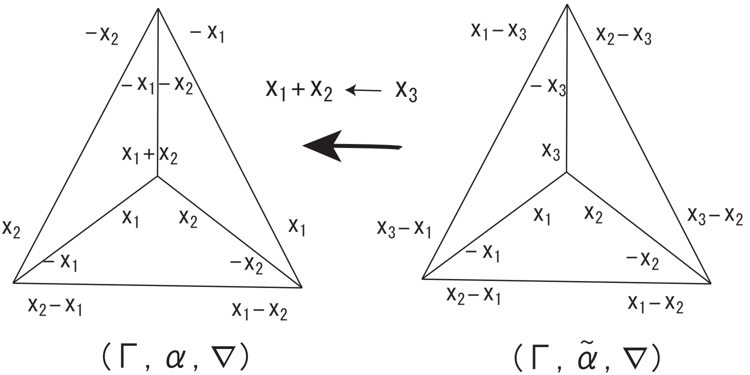

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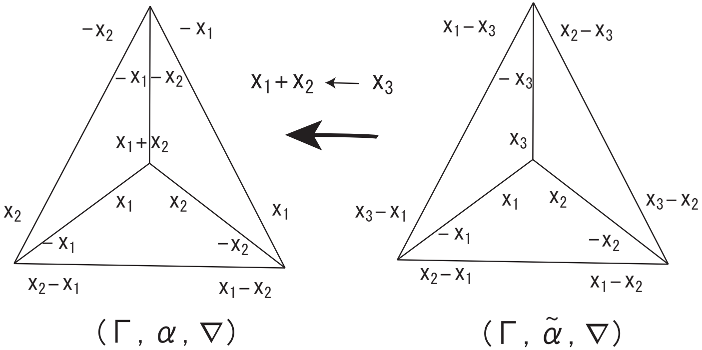
$$\sum_{i=1}^{m} \rho(e_i)\tau_i.$$

Here, e_i *is a normal edge of a facet* F_i *(corresponding to* τ_i*).*

Example of Theorem 1

Compute $H^*(\Gamma, \alpha)$.



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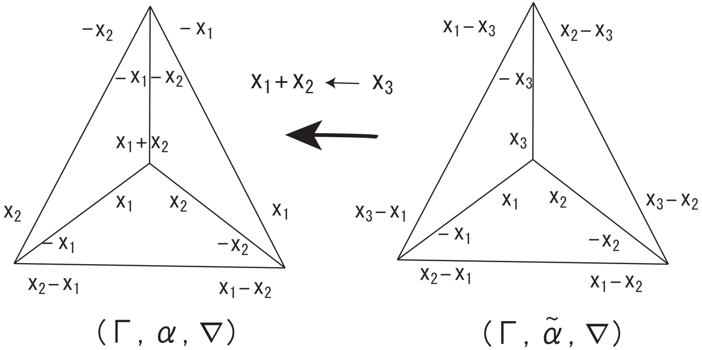
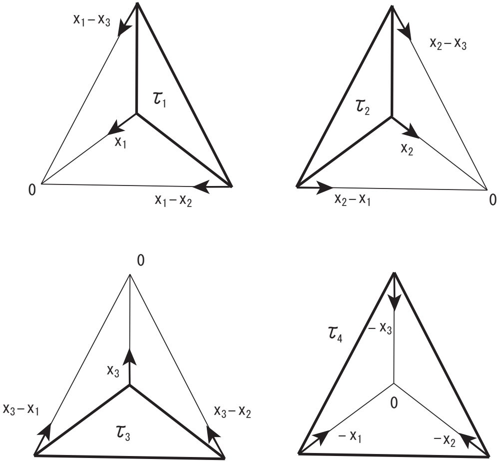


By Maeda-Masuda-Panov's theorem, $H^*(\Gamma, \widetilde{\alpha})$ is isomorphic to

$$\mathbb{Z}[\Gamma, \nabla] = \mathbb{Z}[\tau_1, \tau_2, \tau_3, \tau_4]/\langle \tau_1 \cdots \tau_4 \rangle,$$

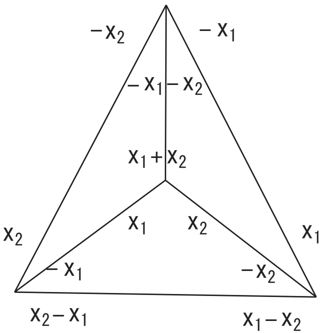
Thom classes $\tau_1, \ldots, \tau_4$ of $(\Gamma, \widetilde{\alpha}, \nabla)$

where



Gale dual of (Γ, α, ∇)

Since the ideal $\mathcal{J}$ is computed by (Γ, ρ, ∇), we compute (Γ, ρ, ∇):



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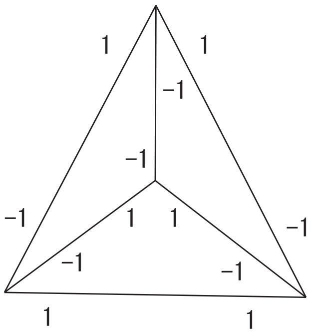

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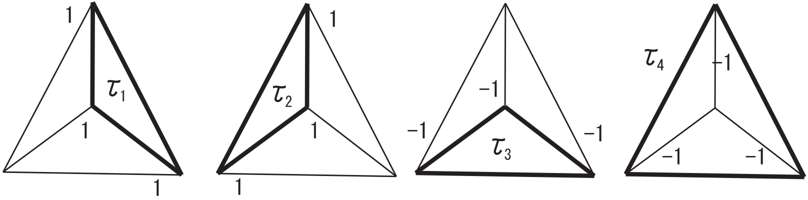

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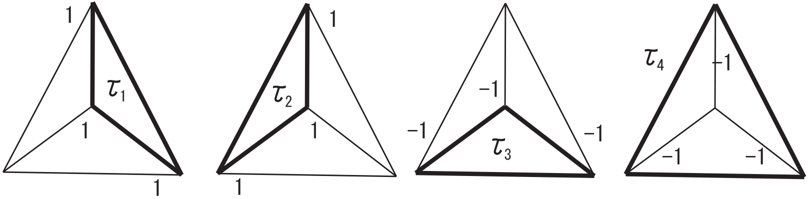

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Therefore, by Theorem 1,

$$H^*(\Gamma, \alpha) \simeq \mathbb{Z}[\Gamma, \nabla]/\langle \tau_1 + \tau_2 - \tau_3 - \tau_4 \rangle$$
$$\simeq \mathbb{Z}[\tau_1, \ldots, \tau_4]\langle \tau_1 \cdots \tau_4, \tau_1 + \tau_2 - \tau_3 - \tau_4 \rangle$$

Meaning of $\sum_{i=1}^{n+1} \rho(e_i)\tau_i$

Note $H^*(\Gamma, \alpha) \simeq H^*(\Gamma, \widetilde{\alpha})/\langle x_1 + x_2 - x_3 \rangle$.

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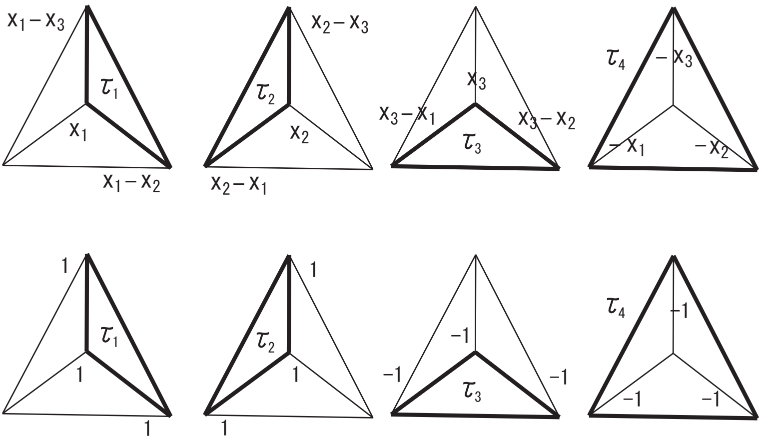
We have $\tau_1 + \tau_2 - \tau_3 - \tau_4 = x_1 + x_2 - x_3$ by



§5 Theorem 2: Group of axial functions and the Gale dual

Put $\mathcal{A}(=\mathcal{A}(\Gamma;(\nabla,C)))$ be the group of axial functions [K. 2019] defined by

$$\mathcal{A}=\{f:V\to\mathbb{Z}^{n+1}\mid \nabla_{pq}(f_p)-f_q=\langle f_q,qp\rangle C(qp)\}$$

where, for $E_p=\{e_1,\ldots,e_{n+1}\}$, $f_p:=f(p)\in\mathbb{Z}^{n+1}=\mathbb{Z}E_p$,

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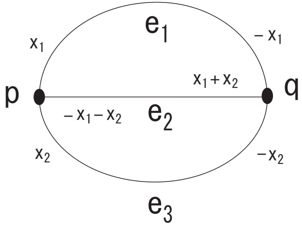
$$C(pq):=(C_{pq}(e_1),\ldots,C_{pq}(e_{n+1})).$$

Example of $\mathcal{A}$

Let (Γ, α, ∇) be



$C : E \to \mathbb{Z}^3$ is defined by $C(pq) = (C_{pq}(e_1), C_{pq}(e_2), C_{pq}(e_3))$ from $\alpha(\nabla_{pq}(e)) - \alpha(e) = C_{pq}(e)\alpha(pq)$.

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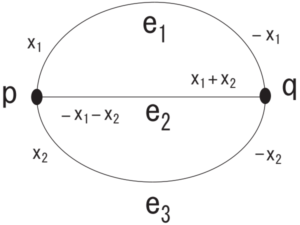
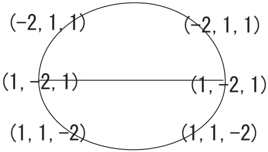


$C : E \to \mathbb{Z}^3$ is defined by $C(pq) = (C_{pq}(e_1), C_{pq}(e_2), C_{pq}(e_3))$ from $\alpha(\nabla_{pq}(e)) - \alpha(e) = C_{pq}(e)\alpha(pq)$. Therefore,



So, we have

$$\begin{aligned}\mathcal{A} &= \{f : \{p, q\} \to \mathbb{Z}^3 \mid \nabla_{e_i}(f_p) - f_q = \langle f_q, \overline{e_i}\rangle C(\overline{e_i})\} \\ &= \{(f_p, f_q) = ((x, y, z), (-x, -y, -z)) \mid x + y + z = 0\} \\ &\simeq \mathbb{Z}^2.\end{aligned}$$

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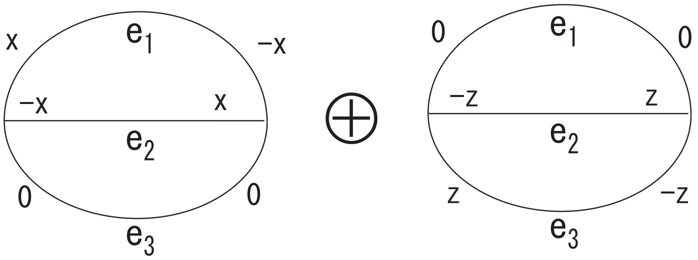
Remark

$\mathcal{A}$ is generated by



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Let (Γ, ρ, ∇) be the Gale dual of (Γ, α, ∇) and $\mathcal{A}$ be the group of axial functions of (Γ, α, ∇). Then, the following theorem holds:

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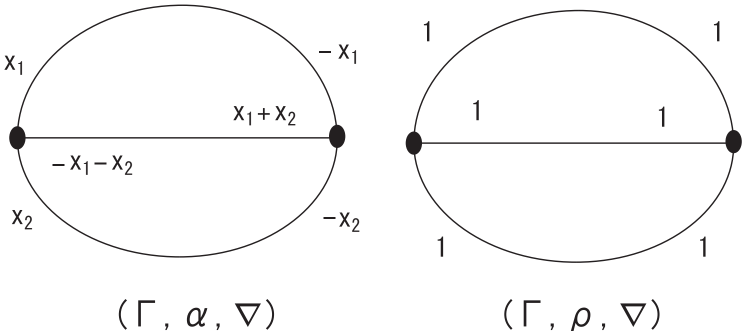
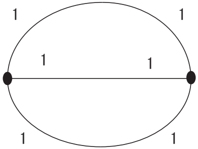
Example of Theorem 2

The Gale dual of (Γ, α, ∇) is



The Gale dual on each vertex



(Γ, ρ, ∇)

is not an element of $\mathcal{A}$, because

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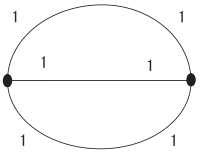

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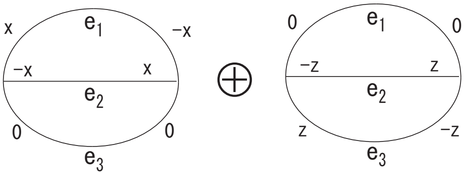

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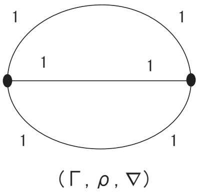

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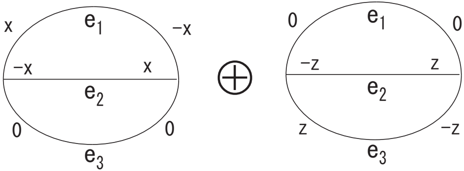

Therefore, by Theorem 2, (Γ, α, ∇) does not extend to the torus graph $(\Gamma, \widetilde{\alpha}, \nabla)$.

Thank you for your attention

Note for zoom 1

Note for zoom 2