Limits of The Burnside Rings and Their Relations

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Let G be a finite group, let $\mathcal{S}(G)$ denote the set of all subgroups of G, and let $\mathcal{F}$ and $\mathcal{G}$ be subsets of $\mathcal{S}(G)$ each of which is closed under taking conjugations and subgroups of its members. For a Mackey functor $M = (M^*, M_*)$ on G, there are the inverse limit $M^*(\mathcal{F})$, the direct limit $M_*(\mathcal{G})$, and the commutative diagram

$$\begin{array}{ccccc} & & M(G) & & \\ & \overset{\mathrm{ind}_{\mathcal{G}}^{G}}{\nearrow} & & \overset{\mathrm{res}_{\mathcal{F}}^{G}}{\searrow} & \\ M_*(\mathcal{G}) & & \xrightarrow[\psi_{\mathcal{G},\mathcal{F}}]{} & & M^*(\mathcal{F}). \end{array}$$

The groups $Q_M(\mathcal{G}, \mathcal{F})$ and $Q_M(G, \mathcal{F})$ are defined by

$$Q_M(\mathcal{G}, \mathcal{F}) = M^*(\mathcal{F})/\psi_{\mathcal{G},\mathcal{F}}(M_*(\mathcal{G})) \;\text{ and }\; Q_M(G, \mathcal{F}) = Q_M(\mathcal{S}(G), \mathcal{F}).$$

For a finite group K, we denote by $A(K)$ the Burnside ring of K. The Burnside ring functor $A = (A^*, A_*)$ on G, where $A^*(H) = A(H) = A_*(H)$ for $H \in \mathcal{S}(G)$, is a Green ring functor on G, and hence a Mackey functor on G in the sense of A. Bak. We are interested in the quotient $Q_A(G, \mathcal{F})$, especially for the case $\mathcal{F} = \mathcal{F}_G := \mathcal{S}(G) \smallsetminus \{G\}$.

Y. Hara and M. Morimoto showed that $Q_A(G, \mathcal{F})$ is a finite group and that in the case where G is nilpotent, $Q_A(G, \mathcal{F}_G)$ is trivial if and only if the order of G is a product of distinct primes (hence G is cyclic).

Let k_G denote the product of (distinct) primes p such that G possesses a normal subgroup of index p. For a prime p, let G^p denote the smallest normal subgroup N of G such that $|G/N|$ is a power of p. Let G^{nil} denote the smallest normal subgroup N of G such that G/N is nilpotent.

Theorem 1. *The following* (1) *and* (2) *hold via canonical isomorphisms.*

(1) $k_G\, Q_A(G, \mathcal{F}_G) = O$ *and* $Q_A(G, \mathcal{F}_G) = \prod_{p|k_G} Q_A(G, \mathcal{F}_G) \otimes \mathbb{Z}_p$ *(p prime).*

(2) $Q_A(G, \mathcal{F}_G) = Q_A(G/G^{\mathrm{nil}}, \mathcal{F}_{G/G^{\mathrm{nil}}}) = \prod_{p|k_G} Q_A(G/G^p, \mathcal{F}_{G/G^p})$ *(p prime).*

Corollary 2. $Q_A(G, \mathcal{F}_G) = O$ *if and only if* G/G^{nil} *is a cyclic group such that* $|G/G^{\mathrm{nil}}|$ *is a product of distinct primes.*

We will also report computations of $Q_A(G, \mathcal{F}_G)$ for some abelian p-groups G.