Inverse problems and theory of reproducing kernels – numerical experiments(abstract)

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1 Formulations and solutions

Here, for the three inverse problems previously shown by Prof. Saito in his abstract, we present formurations and solutions for those problems using Tikhonov regularization.

1) Inverse source problem in the Poisson equation([1])
We obtained very and surprisingly simple approximate solutions for the Poisson equation, for any $L_2(\mathbf{R}^n)$ function g,

$$\Delta u = g \quad \text{on} \quad \mathbf{R}^n \tag{1.1}$$

in the class of the functions of the s order Sobolev Hilbert space H^s on the whole real space $\mathbf{R}^n (n \geq 1, s \geq 2, s > n/2)$.

We shall use the n order Sobolev Hilbert space H^n comprising functions F on $\mathbf{R}^n$ with the norm (Here, of course, $r_1 + r_2 + \cdots + r_n = \nu$.)

$$\|F\|_{H_S}^2 = \sum_{\nu=0}^{n} {}_nC_\nu \sum_{r_1,r_2,\ldots,r_n \geq 0} \frac{\nu!}{r_1!r_2!\cdots r_n!} \int_{\mathbf{R}^n} \left(\frac{\partial^\nu F(x)}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_n^{r_n}} \right)^2 dx. \tag{1.2}$$

This Hilbert space admits the reproducing kernel

$$K(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1+|\xi|^2)^n} e^{i(x-y)\cdot\xi} d\xi \tag{1.3}$$

as we see easily by using Fourier's transform.Note that the Sobolev Hilbert space H^s admitting the reproducing kernel (1.3) for $n = s$ can be defined for

any positive number s in term of Fourier integrals $\hat{F}$ of F

$$\hat{F}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-i\xi \cdot x} F(x) dx$$

as follows:

$$\|F\|_{H^s}^2 = \int_{\mathbf{R}^n} |\hat{F}(\xi)|^2 (1 + |\xi|^2)^s d\xi$$

for any $s > n/2$.

Under these conditions our formulations and results are stated as follows:

Theorem 1 *Let $n \geq 1, s \geq 2$ and $s > n/2$. For any function $g \in L_2(\mathbf{R}^n)$ and for any $\lambda > 0$, the best approximate function $F_{\lambda,s,g}^*$ in the sense*

$$\inf_{F \in H^s} \left\{ \lambda \|F\|_{H^s}^2 + \|g - \Delta F\|_{L_2(\mathbf{R}^n)}^2 \right\} = \lambda \|F_{\lambda,s,g}^*\|_{H^s}^2 + \|g - \Delta F_{\lambda,s,g}^*\|_{L_2(\mathbf{R}^n)}^2 \tag{1.4}$$

exists uniquely and $F_{\lambda,s,g}^*$ is represented by

$$F_{\lambda,s,g}^*(x) = \int_{\mathbf{R}^n} g(\xi) Q_{\lambda,s}(\xi - x) d\xi \tag{1.5}$$

for

$$Q_{\lambda,s}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{-|p|^2 e^{-ip \cdot (\xi - x)} dp}{\lambda(|p|^2 + 1)^s + |p|^4}. \tag{1.6}$$

If, for $F \in H^s$ we consider the solution $u_F(x)$: $\Delta u_F(x) = F(x)$ and we take $u_F(\xi)$ as g, then we have the favourable result: as $\lambda \to 0$

$$F_{\lambda,s,g}^* \to F, \tag{1.7}$$

uniformly.

2) The problem in the heat conduction([2]); that is, from some heat $u(x,t)$ observation at a time t, look for the initial heat $u(x,0)$.

We gave simple approximate real inversion formulas for the Gaussian convolution (the Weierstrass transform)

$$u_F(x,t) = (L_t F)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} F(\xi) \exp \left\{ -\frac{|\xi - x|^2}{4t} \right\} d\xi \tag{1.8}$$

for the functions of the s order Sobolev Hilbert space H^s on the whole real space $\mathbf{R}^n (n \geq 1, s > n/2)$. This integral transform which represents the solution $u(x,t)$ of the heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{on} \quad \mathbf{R}^n \times \{t > 0\} \quad (u(x,0) = F(x) \quad \text{on} \quad \mathbf{R}^n). \tag{1.9}$$

In this problem we can set same norm and reproducing kernel as (1.2) and (1.3).

Under those situations our formulations and results are stated as follows:

Theorem 2 *For any function $g \in L_2(\mathbf{R}^n)$ and for any $\lambda > 0$, the best approximate function $F^*_{\lambda,s,g}$ in the sense*

$$\inf_{F \in H^s} \left\{ \lambda \|F\|^2_{H^s} + \|g - u_F(\cdot, t)\|^2_{L_2(\mathbf{R}^n)} \right\}$$

$$= \lambda \|F^*_{\lambda,s,g}\|^2_{H^s} + \|g - u_{F^*_{\lambda,s,g}}(\cdot, t)\|^2_{L_2(\mathbf{R}^n)} \tag{1.10}$$

exists uniquely and $F^*_{\lambda,s,g}$ is represented by

$$F^*_{\lambda,s,g}(x) = \int_{\mathbf{R}^n} g(\xi) Q_{\lambda,s}(\xi - x) d\xi \tag{1.11}$$

for

$$Q_{\lambda,s}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{e^{-ip \cdot (\xi - x)} dp}{\lambda(|p|^2 + 1)^s e^{|p|^2 t} + e^{-|p|^2 t}}. \tag{1.12}$$

If, for $F \in H^s$ we consider the output $u_F(x,t)$ and we take $u_F(\xi, t)$ as g, then we have the favourable result: as $\lambda \to 0$

$$F^*_{\lambda,s,g} \to F, \tag{1.13}$$

uniformly.

3) Real inversion formulas for the Laplace transform([3])

We obtained a very natural and numerical real inversion formula of the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt} F(t) dt, \quad p > 0 \tag{1.14}$$

for functions F of some natural function space. The inversion of the Laplace transform is, in general, given by a complex form, however, we are interested in and are requested to obtain its real inversion in many practical problems. However, the real inversion will be very involved and one might think that its real inversion will be essentially involved, because we must catch "analyticity" from the real or discrete data.

We shall introduce the simple reproducing kernel Hilbert space (RKHS) H_K comprised of absolutely continuous functions F on the positive real line $\mathbf{R}^+$ with finite norms

$$\left\{ \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt \right\}^{1/2} \quad (\ F(0) = 0\ ). \tag{1.15}$$

This Hilbert space admits the reproducing kernel

$$K(t, t') = \int_0^{\min(t,t')} \xi e^{-\xi} d\xi. \tag{1.16}$$

Then we see that

$$\int_0^\infty |(\mathcal{L}F)(p)p|^2 dp \le \frac{1}{2}\|F\|_{H_K}^2; \tag{1.17}$$

that is, the linear operator on H_K,$(\mathcal{L}F)(p)p$ into $L_2(\mathbf{R}^+, dp) = L_2(\mathbf{R}^+)$ is bounded([4]). For the reproducing kernel Hilbert spaces H_K satisfying (1.17), we can find some general spaces.Therefore, from the general theory in [4], we obtain

Theorem 3 *([4]). For any $g \in L_2(\mathbf{R}^+)$ and for any $\alpha > 0$, the best approximation $F_{\alpha,g}^*$ in the sense*

$$\inf_{F \in H_K} \left\{ \alpha \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F)(p)p - g\|_{L_2(\mathbf{R}^+)}^2 \right\}$$

$$= \alpha \int_0^\infty |F_{\alpha,g}^{*\prime}(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F_{\alpha,g}^*)(p)p - g\|_{L_2(\mathbf{R}^+)}^2 \tag{1.18}$$

exists uniquely and we obtain the representation

$$F_{\alpha,g}^*(t) = \int_0^\infty g(\xi) \left(\mathcal{L}K_\alpha(\cdot,t)\right)(\xi)\xi d\xi. \tag{1.19}$$

Here, $K_\alpha(\cdot,t)$ is determined by the functional equation

$$K_\alpha(t,t') = \frac{1}{\alpha}K(t,t') - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}K_t)(p)p)_{L_2(\mathbf{R}^+)} \tag{1.20}$$

for $K_{\alpha,t'} = K_\alpha(\cdot,t')$ and $K_t = K(\cdot,t)$.

We shall look for the approximate inversion $F_{\alpha,g}^*(t)$ by using (1.19). For this purpose, we take the Laplace transfrom of (1.20) in t and change the variables t and t' as in

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi) = \frac{1}{\alpha}(\mathcal{L}K(\cdot,t'))(\xi) - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}(\mathcal{L}K_t)(p)p))(\xi))_{L_2(\mathbf{R}^+)}. \tag{1.21}$$

Note that

$$K(t,t') = \left\{ \begin{array}{ll} -te^{-t} - e^{-t} + 1 & \text{for} \quad t \le t' \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for} \quad t \ge t'. \end{array} \right.$$

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi) = e^{-t'p}e^{-t'}\left[\frac{-t'}{p(p+1)} + \frac{-1}{p(p+1)^2}\right] + \frac{1}{p(p+1)^2}. \tag{1.22}$$

$$\int_0^\infty e^{-qt'}(\mathcal{L}K(\cdot,t'))(p)dt' = \frac{1}{pq(p+q+1)^2}. \tag{1.23}$$

Therefore, by setting $(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t)$, which is needed in (1.19), we obtain the Fredholm integral equation of the second type

$$\alpha H_\alpha(\xi,t) + \int_0^\infty H_\alpha(p,t)\frac{1}{(p+\xi+1)^2}dp = -\frac{e^{-t\xi}e^{-t}}{\xi+1}\left(t + \frac{1}{\xi+1}\right) + \frac{1}{(\xi+1)^2}. \tag{1.24}$$

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2   Some examples of numerical experiments

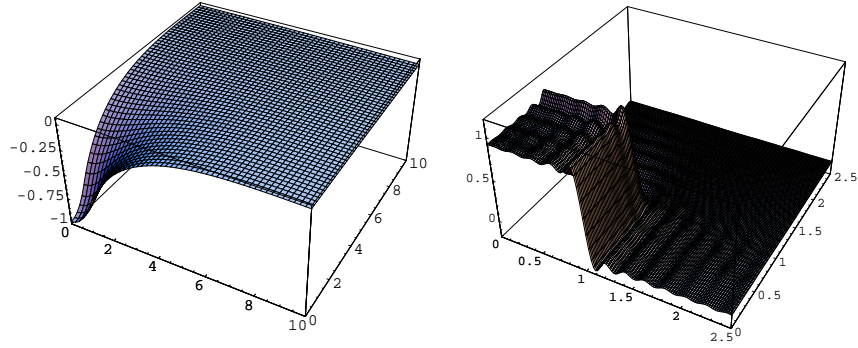**1) Inverse source problem in the Poisson equation**([1])

Figure 1: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Delta F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^{-2}$.

This numerical result shows that the new method ((1.5),(1.6)) is working effectively and is useful.
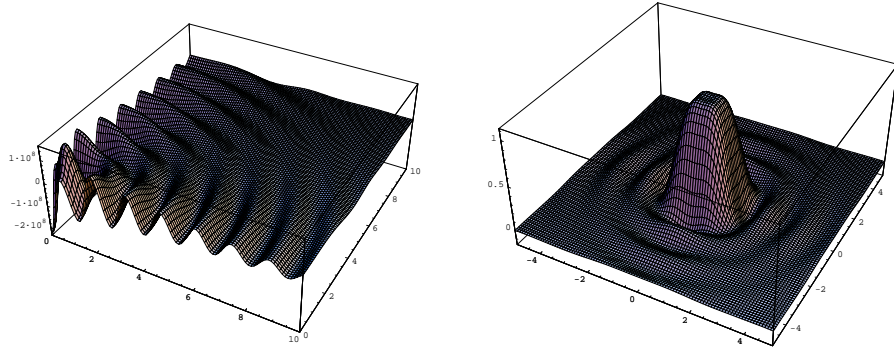
2) The problem in the heat conduction([2])

Figure 2: For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,s,g}(x_1, x_2)$ and $u_{F^*_{\lambda,s,g}}(x_1, x_2; t)$ for $t = 1, s = 2, \lambda = 10^{-22}$.

The results of this numerical experiment prove the usefulness and correctness of our method.

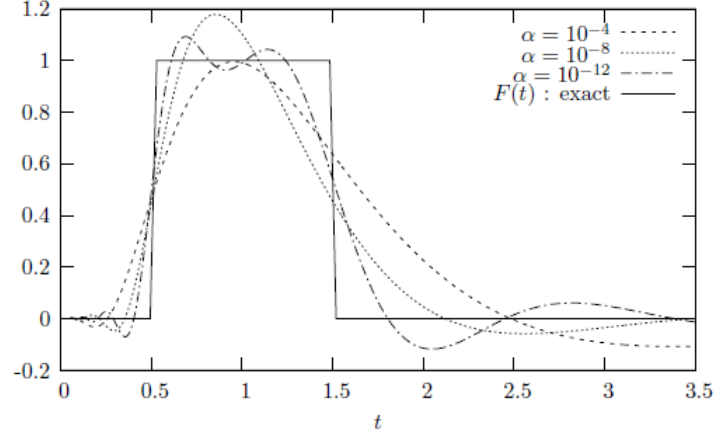**3) Real inversion formulas for the Laplace transform([3])**

Figure 3: For $F(t) = \chi(t, [1/2, 3/2])$, the characteristic function and for $\alpha = 10^{-1}, 10^{-4}, 10^{-8}, 10^{-12}, 10^{-16}$.
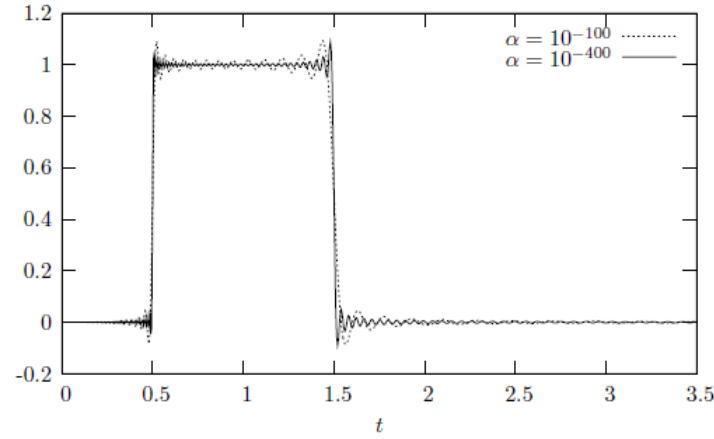

Figure 4: For $F(t) = \chi(t, [1/2, 3/2])$, the characteristic function and for $\alpha = 10^{-1}, 10^{-4}, 10^{-8}, 10^{-12}, 10^{-16}$.

The results of these numerical experiments show that our method is effective even when there are jumps in the target function, and in Figure 4 we use a high-precision numerical algorithm developed by our collaborator Professor Fujiwara.

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References

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[3] T. Matsuura, A. Al-Shuaibi and H. Fujiwara, Numerical Real Inversion Formulas of the Laplace Transform by using a Fredholm integral equation of the second kind, Journal of Analysis and Applications, **5**(2007), 123–136.

[4] S. Saitoh, Approximate real inversion formulas of the Laplace transform, Far East J. Math. Sci.,**11**(2003), 53–64.