Numerical analysis of constrained total variation flows and its application to the Kobayashi–Warren–Carter model*

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1 Introduction

A general constrained total variation (TV) flow for $u\colon \Omega \times [0,T) \to M$ is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\pi_u \left(-\nabla \cdot \dfrac{\nabla u}{|\nabla u|} \right) & \text{in } \Omega \times (0,T), \\ \dfrac{\nabla u}{|\nabla u|} \cdot \nu^\Omega = 0 & \text{on } \partial\Omega \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^k$ $(k \geq 1)$ is a bounded region with Lipschitz boundary $\partial\Omega$, M a manifold embedded into $\mathbb{R}^l$ $(l \geq 1)$, $u_0\colon \Omega \to M$ an initial datum, π_p the orthogonal projection from the tangent space $T_p\mathbb{R}^l$ $(= \mathbb{R}^l)$ to the tangent space T_pM $(\subset \mathbb{R}^l)$ at $p \in M$, ν^Ω the unit outward normal vector of $\partial\Omega$, and $T > 0$. If π_u is absent, (1) is the standard vectorial total variation flow of the isotropic total variation of vector-valued maps:

$$\mathbf{TV}(u) \coloneqq \int_\Omega |\nabla u|_{\mathbb{R}^{k\times l}} \, \mathrm{d}x$$

The introduction of π_u means that we restrict the gradient of total variation so that u always takes value in M.

Constrained TV flow (1) appears in diverse fields. The first application of this flow appears in [4], where the authors considered the two-dimensional sphere S^2 as the target manifold to denoise color images while preserving brightness. This system, where the target manifold is the space $SO(3)$ of all three-dimensional rotations, appears in the Kobayashi–Warren–Carter model [3, 2], a basic prototype of the continuum model of the time evolution of grain boundaries in a crystal.

Despite its broad applicability, there is still much room for research in the mathematical and numerical analysis of constrained TV flows. This talk aims to develop a new numerical scheme for spatially discrete constrained total variation flows based on the exponential map and the minimizing movement scheme.

2 Spatially discrete model and numerical scheme

2.1 Finite-dimensional Hilbert spaces and discrete constrained TV flows

Let Δ be a finite set of indices and let Ω be a bounded region in $\mathbb{R}^k$. A family $\Omega_\Delta = \{\Omega_\alpha\}_{\alpha\in\Delta}$ of subsets of Ω is a rectangular partition of Ω if Ω_Δ satisfies the following three conditions.

(i) $\mathscr{L}^k \left(\Omega \setminus \bigcup_{\alpha\in\Delta} \Omega_\alpha \right) = 0$.

(ii) $\mathscr{L}^k(\Omega_\alpha \cap \Omega_\beta) = 0$ for $\alpha \neq \beta$, $(\alpha, \beta) \in \Delta \times \Delta$; here $\mathscr{L}^k$ denotes the Lebesgue measure.

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(iii) For each $\alpha \in \Delta$, there exists a rectangular region $R_\alpha \in \mathbb{R}^k$ such that $\Omega_\alpha = R_\alpha \cap \Omega$ holds. Here, by a rectangular region, we mean for some $a_j < b_j$ that

$$R_\alpha = \{x = (x_1, \dots, x_k) \mid a_j < x_j < b_j,\ j = 1, 2, \dots, k\}.$$

We denote by $e(\Delta)$ the set of edges associated with Δ defined as

$$e(\Delta) := \{\{\alpha, \beta\} \subset \Delta \mid \mathcal{H}^{k-1}(\partial\Omega_\alpha \cap \partial\Omega_\beta) \neq 0,\ \alpha \neq \beta\},$$

where $\mathcal{H}^{k-1}$ denotes the $(k-1)$-dimensional Hausdorff measure.

Subsequently, we fix the rectangular partition Ω_Δ of Ω. Then, define the space of piecewise $\mathbb{R}^l$-valued constant functions as

$$H_\Delta := \left\{ u = \sum_{\alpha \in \Delta} u^\alpha 1_{\Omega_\alpha} \;\middle|\; u^\alpha \in \mathbb{R}^l,\ \alpha \in \Delta \right\}.$$

We can regard that H_Δ is a closed subspace of the Hilbert space $L^2(\Omega; \mathbb{R}^l)$ with an inner product

$$\langle X, Y \rangle_{H_\Delta} := \langle X, Y \rangle_{L^2(\Omega; \mathbb{R}^l)}.$$

The discrete total variation functional $\mathbf{TV}_\Delta \colon H_\Delta \to \mathbb{R}$ associated with Ω_Δ is defined as follows:

$$\mathbf{TV}_\Delta(u) := \sum_{\{\alpha,\beta\} \in e(\Delta)} \|u^\alpha - u^\beta\|_{\mathbb{R}^l} \mathcal{H}^{k-1}(\partial\Omega_\alpha \cap \partial\Omega_\beta).$$

Moreover, we define the space of piecewise M-valued constant functions as

$$M_\Delta := \left\{ u = \sum_{\alpha \in \Delta} u^\alpha 1_{\Omega_\alpha} \;\middle|\; u^\alpha \in M,\ \alpha \in \Delta \right\}.$$

We finally introduce the orthogonal projection $P_u \colon H_\Delta \to M_\Delta$ at $u \in M_\Delta$ by

$$P_u X(x) := \pi_{u(x)}(X(x)) \quad \text{for a.e.} \quad x \in \Omega.$$

Definition 2.1. Let $u_0 \in M_\Delta$ and $I := [0, T)$. A map $u \in W^{1,2}(I; M_\Delta)$ is said to be a solution to the discrete model of (1) if u satisfies

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} \in -P_{u(t)} \partial \mathbf{TV}_\Delta(u(t)) \quad \text{for a.e. } t \in (0, T), \\ u|_{t=0} = u_0. \end{cases} \tag{2}$$

2.2 Numerical scheme

Let τ be the time step and define $t^n := n\tau$. According to the minimizing movement scheme, from solution $u_\tau^{(n-1)}$ at the current time, we want to define the solution $u_\tau^{(n)}$ at the next time as the minimizer to the following problem:

$$\text{minimize} \quad \Phi^\tau(u; u_\tau^{(n-1)}) := \tau \mathbf{TV}_\Delta(u) + \frac{1}{2}\|u - u_\tau^{(n-1)}\|_{H_\Delta}^2 \quad \text{subject to} \quad u \in M_\Delta.$$

However, it is not easy to solve this problem since it is a non-smooth Riemannian constraint optimization problem. To overcome this difficulty, we localize the energy by employing the exponential map. Thanks to the exponential map in M, each $u \in M_\Delta$ can be written as a $u_\tau^{(n-1)}$ and $X \in T_{u_\tau^{(n-1)}} M_\Delta$ pair: $u = \mathrm{Exp}_{u_\tau^{(n-1)}}(X)$, where $\mathrm{Exp}_{u_\tau^{(n-1)}} \colon T_{u_\tau^{(n-1)}} M_\Delta \to M_\Delta$ is defined as

$$X(x) \longmapsto \exp_{u_\tau^{(n-1)}(x)}(X(x)) \quad \text{for a.e.} \quad x \in \Omega.$$

Here, $T_u M_\Delta$ denotes the tangent space of M_Δ at $u \in M_\Delta$, and $\exp_x$ is the exponential map of the Riemannian manifold M. Since $\mathrm{Exp}_{u_\tau^{(n-1)}}(X) = u_\tau^{(n-1)} + X + o(X)$, we ignore the term $o(X)$ and insert $u = u_\tau^{(n-1)} + X$ into $\Phi^\tau(u, u_\tau^{(n-1)})$ to obtain the localized energy

$$\Phi_{\mathrm{loc}}^\tau(X; u_\tau^{(n-1)}) := \tau \mathbf{TV}_\Delta(u_\tau^{(n-1)} + X) + \frac{1}{2}\|X\|_{H_\Delta}^2, \quad X \in H_\Delta.$$

As a result, we obtain the following modified minimizing movement scheme.

(i) From solution $u_\tau^{(n-1)}$ at the current time, we compute the unique minimizer $X_\tau^{(n-1)}$ of the following problem:

$$\text{minimize} \quad \Phi_{\text{loc}}^\tau(X; u_\tau^{(n-1)}) \quad \text{subject to} \quad X \in T_{u_\tau^{(n-1)}} M_\Delta.$$

(ii) Set $u_\tau^{(n)} \coloneqq \mathrm{Exp}_{u_\tau^{(n-1)}}(X_\tau^{(n-1)})$.

Concerning the above constructed numerical scheme, we obtain the following results.

Theorem 2.2 (Energy dissipation)**.** *Let M be a C^2-compact manifold embedded into $\mathbb{R}^l$, $I \coloneqq [0,T)$, $u_0 \in M_\Delta$ be an initial datum, $\tau > 0$ be a step size, and $\{u_\tau^{(n)}\}_n$ be a sequence generated by the modified minimizing movement scheme. If τ is sufficiently small, then $\mathbf{TV}_\Delta(u_\tau^{(n+1)}) \leq \mathbf{TV}_\Delta(u_\tau^{(n)})$ holds.*

Namely, the proposed scheme inherits the property that the total variation dissipates along with the flow.

Theorem 2.3 (Error estimate)**.** *Let M be a path-connected and C^2-compact manifold embedded into $\mathbb{R}^l$, $I \coloneqq [0,T)$, $u_0 \in M_\Delta$ be an initial datum, and $\tau > 0$ be a step size. Denote by u the solution of the discrete model* (2) *and by $\{u_\tau^{(n)}\}_n$ the sequence obtained by the modified minimizing movement scheme. Then, there exist constants C_0, C_1, and C_2 independent of u and τ such that the following error estimate holds:*

$$\|u_\tau(t^n) - u_\tau^{(n)}\|_{H_\Delta}^2 \leq t^n \mathrm{e}^{C_0 t^n} \left(C_1 \tau + C_2 \tau^2 \right), \quad n = 0, 1, \ldots.$$

This theorem implies that the accuracy of the proposed scheme is the order of $\sqrt{\tau}$.

In the talk, we will present the details of the derivation of the above numerical scheme and the idea of the proof of two theorems. We will also show numerical results for constrained total variation flows, where the target manifolds are S^2 and $SO(3)$. Finally, we introduce the spatially discrete Kobayashi–Warren–Carter energy and numerically study the solution's asymptotic behavior to the gradient flow of its energy compared with the Ambrosio–Tortorelli energy.

The contents of the constrained total variation flow are summarized in our paper [1].

References

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