Reconstruction of the defect by the enclosure method for inverse problems of the magnetic Schrödinger operator

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This study is based on the paper [1]. We give the formula to extract the position and the shape of the defect D generated in the object (conductor) Ω from the observation data on the boundary $\partial\Omega$ for the magnetic Schrödinger operator by using the enclosure method proposed by Ikehata [2]. We show a reconstruction formula of the convex hull of the defect D from the observed data, assuming certain higher regularity for the potentials of the magnetic Schrödinger operator, under the Dirichlet condition or the Robin condition on the boundary ∂D in the two and three dimensional case.

Let $\Omega \subset R^n (n = 2, 3)$ be a bounded domain where the boundary $\partial\Omega$ is C^2 and let D be an open set satisfying $\overline{\mathrm{D}} \subset \Omega$ and $\Omega \setminus \overline{\mathrm{D}}$ is connected. The defect D consists of the union of disjoint bounded domains $\{D_j\}_{j=1}^n$, where the boundary of D is Lipschitz continuous. First, we define the DN map for the magnetic Schrödinger equation with no defect D in Ω. Here, let $D_A^2 u := \sum_{j=1}^n D_{A,j}(D_{A,j}u)$, where $D_{A,j} := \frac{1}{i}\partial_j + A_j$ and $A = (A_1, A_2, \cdots, A_n)$.

Definition 1. Suppose $q \in L^\infty(\Omega), q \geq 0, A \in C^1(\overline{\Omega}, R^n)$. For a given $f \in H^{1/2}(\partial\Omega)$, we say $u \in H^1(\Omega)$ is a weak solution to the following boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} D_A^2 u + qu = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

if $u = f$ on $\partial\Omega$ and u satisfies

$$\int_\Omega (D_A u) \cdot \overline{D_A \varphi} + qu\overline{\varphi}\, dx = 0$$

for any $\varphi \in H^1(\Omega)$ such that $\varphi|_{\partial\Omega} = 0$. Here, $\overline{\varphi}$ is the complex conjugate of φ.

The DN map $\Lambda_{q,A}$ is defined as follows.

Definition 2. (Weak formulation of DN map)
The DN map $\Lambda_{q,A} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is defined as follows by the duality:

$$\langle \Lambda_{q,A} f\ ,\ \overline{g} \rangle = \int_\Omega (D_A u) \cdot \overline{D_A v} + qu\overline{v}\, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where $u \in H^1(\Omega)$ is the weak solution of (1.1) and $v \in H^1(\Omega)$ is any function satisfying $v|_{\partial\Omega} = g$.

We define the weak solution of the magnetic Schrödinger equation with a defect D in Ω under the Robin boundary condition on ∂D.

Definition 3. (Robin case)
Suppose $q \in L^\infty(\Omega \setminus \overline{D}), q \geq 0, \lambda \in C^1(\partial D), \lambda \geq 0$ and $A \in C^1(\overline{\Omega \setminus \overline{D}}, R^n)$. Let ν is the outward unit normal vector to $\Omega \setminus \overline{D}$. For a given $f \in H^{1/2}(\partial\Omega)$, we say $u \in H^1(\Omega \setminus \overline{D})$ is a weak solution to the following value problem for the magnetic Schrödinger equation

$$\begin{cases} D_A^2 u + qu = 0 \text{ in } \Omega \setminus \overline{D}, \\ \nu \cdot (\nabla + iA)u + \lambda u = 0 \text{ on } \partial D, \\ u = f \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

if $u = f$ on $\partial\Omega$ and u satisfies

$$\int_{\Omega \setminus \overline{D}} (D_A u) \cdot \overline{D_A \varphi} + qu\overline{\varphi}\, dx + \int_{\partial D} \lambda u \overline{\varphi}\, dS = 0$$

for any $\varphi \in H^1(\Omega \setminus \overline{D})$ such that $\varphi|_{\partial\Omega} = 0$.

The DN map $\Lambda^{(R)}_{q,A,D}$ is defined as follows.

Definition 4. (DN map of the Robin case)
The DN map $\Lambda^{(R)}_{q,A,D} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is defined as follows by the duality:

$$\langle \Lambda^{(R)}_{q,A,D} f \ , \ \overline{g} \rangle = \int_{\partial D} \lambda u \overline{v} \, dS + \int_{\Omega \setminus \overline{D}} (D_A u) \cdot \overline{D_A v} + q u \overline{v} \, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where $u \in H^1(\Omega \setminus \overline{D})$ is the weak solution of (1.3) and $\varphi \in H^1(\Omega \setminus \overline{D})$ is any function $\varphi|_{\partial\Omega} = g$.
In the special case $\lambda = 0$, we denote $\Lambda^{(N)}_{q,A,D}$ instead of $\Lambda^{(R)}_{q,A,D}$.

Remark 1. The weak solution of the magnetic Schrödinger equation with a defect D in Ω under the Dirichlet boundary condition on ∂D and the DN map $\Lambda^{(D)}_{q,A,D}$ can be defined in a similar way.

Next, we introduce an indicator function that plays an important role in the enclosure method. We denote by S^{n-1} the set of n-dimensional unit vectors ($n = 2, 3$). For a given $\omega \in S^{n-1}$, we can take an orthogonal unit vector $\omega^{\perp} \in S^{n-1}$, namely $\omega \cdot \omega^{\perp} = 0$. Then we can construct a solution $v_\tau(x;\omega) := e^{\tau x \cdot (\omega + i\omega^{\perp})}(1 + r_\tau(x;\omega))$ of $D_A^2 v + qv = 0$, where $r_\tau(x;\omega)$ is chosen suitably associated with the parameter $\tau \in R$. This solution is called the complex geometrical optics solutions.

Definition 5. (Indicator function)
Let $t, \tau \in \mathbb{R}$. Then, the indicator function $I_\omega(\tau;t)$ is defined as follows.

$$I^{(R)}_\omega(\tau;t) := \langle (\Lambda_{q,A} - \Lambda^{(R)}_{q,A,D})(e^{-\tau t} v_\tau(x;\omega)), \overline{e^{-\tau t} v_\tau(x;\omega)} \rangle$$

Here, $\overline{v_\tau}$ is the complex conjugate of v_τ. In the special case $\lambda = 0$, we denote $\Lambda^{(N)}_{q,A,D}$ instead of $\Lambda^{(R)}_{q,A,D}$. Also, $I^{(D)}_\omega(\tau;t)$ can be defined by $\Lambda^{(D)}_{q,A,D}$. We define the support function $h_D(\omega)$ as follows :

$$h_D(\omega) = \sup_{x \in D} x \cdot \omega, \ \ \omega \in S^{n-1}.$$

Then it is well-known that the convex hull $\mathrm{conv}(D)$ of D is obtained as follows.

$$\mathrm{conv}(D) := \cap_{\omega \in S^{n-1}} \{ x \in R^n | x \cdot \omega < h_D(\omega) \}.$$

Since the indicator function $I_\omega(\tau;t)$ is determined from the DN map, if the support function $h_D(\omega)$ is obtained from the indicator function $I_\omega(\tau;t)$, the convex hull $\mathrm{conv}(D)$ of inclusion D can be reconstructed from the observation data on boundary $\partial\Omega$. Now, we give the formula of the reconstruction of the support function from the indicator function under a certain smallness condition for the vector potential A.

Theorem 1. *Suppose ∂D is Lipschitz continuous. Let $n = 2, 3, q \in H^2(\Omega), q \geq 0, A \in H^3(\Omega)$ and $C(\Omega)\|A\|_{H^2(\Omega)} \leq \frac{1}{2}$. Then, we have*

$$\lim_{\tau \to \infty} \frac{\log |I^{(D)}_w(\tau;0)|}{2\tau} = h_D(w), \ \lim_{\tau \to \infty} \frac{\log |I^{(N)}_w(\tau;0)|}{2\tau} = h_D(w),$$

for any $\omega \in S^{n-1}$. Here, the constant $C(\Omega)$ depends only on Ω.

For a given $\omega \in S^{n-1}$, we furthermore assume the following condition $(D)_\omega$ for the Robin case.

$(D)_\omega$**:** Suppose ∂D is C^2 and the set $T(\omega) := \{ x \in \overline{D} \mid h_D(\omega) - x \cdot \omega = 0 \}$ consists of only one point $x_0 \in \partial D$. Furthermore, we assume that in the neighborhood of x_0 the boundary ∂D can be expressed as $y = f(s), |s| < \epsilon, s \in R^{n-1}$, and there exists $K_0, K_1 > 0, m_\omega \geq 2$ such that

$$K_0 |s|^{m_w} \leq f(s) \leq K_1 |s|^{m_w} \quad (|s| < \epsilon).$$

Theorem 2. (*Robin case*) *Suppose $\lambda \neq 0, \lambda \geq 0$ and $\lambda \in C^1(\partial D)$. Let $n = 2, 3, q \in H^2(\Omega), q \geq 0, A \in H^3(\Omega)$ and $C(\Omega)\|A\|_{H^2(\Omega)} \leq \frac{1}{2}$. We assume that the condition $(D)_\omega$ holds as $2 \leq m_w < 3$ for some $\omega \in S^{n-1}$. Then, we have*

$$\lim_{\tau \to \infty} \frac{\log |I^{(R)}_\omega(\tau;0)|}{2\tau} = h_D(\omega).$$

References

[1] K. Kurata and R. Yamashita, *Reconstruction of the defect by the enclosure method for inverse problems of the magnetic Schrödinger operator,* (submitted).

[2] M. Ikehata, *Reconstruction of the support function for inclusion from boundary measurements,* J. Inv. Ill-Posed Problems, **8** (2000), pp. 367-378.