Reconstruction of the defect by the enclosure method for inverse problems of the magnetic Schrödinger operator

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This study is based on the paper [1]. We give the formula to extract the position and the shape of the defect $D$ generated in the object (conductor) $\Omega$ from the observation data on the boundary $\partial \Omega$ for the magnetic Schrödinger operator by using the enclosure method proposed by Ikehata [2]. We show a reconstruction formula of the convex hull of the defect $D$ from the observed data, assuming certain higher regularity for the potentials of the magnetic Schrödinger operator, under the Dirichlet condition or the Robin condition on the boundary $\partial D$ in the two and three dimensional case.

Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ be a bounded domain where the boundary $\partial \Omega$ is $C^2$ and let $D$ be an open set satisfying $\mathring{D} \subset \Omega$ and $\Omega \setminus \mathring{D}$ is connected. The defect $D$ consists of the union of disjoint bounded domains $\{D_j\}_{j=1}^n$, where the boundary of $D$ is Lipschitz continuous. First, we define the DN map for the magnetic Schrödinger equation with no defect $D$ in $\Omega$. Here, let $D_j a u := \sum_{j=1}^n D_{A,j} (D_{A,j} u)$, where $D_{A,j} := \frac{1}{\lambda_j} \partial_j + A_j$ and $A = (A_1, A_2, \cdots, A_n)$.

**Definition 1.** Suppose $q \in L^\infty(\Omega), q \geq 0, A \in C^1(\mathring{\Omega}, \mathbb{R}^n)$. For a given $f \in H^{1/2}(\partial \Omega)$, we say $u \in H^1(\Omega)$ is a weak solution to the following boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} D_A^2 u + qu = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial \Omega, \end{cases} \quad (1.1)$$

if $u = f$ on $\partial \Omega$ and $u$ satisfies

$$\int_{\Omega} (D_A u) \cdot \overline{D_A \varphi} + qu \overline{\varphi} \, dx = 0$$

for any $\varphi \in H^1(\Omega)$ such that $\varphi|_{\partial \Omega} = 0$. Here, $\overline{\varphi}$ is the complex conjugate of $\varphi$.

The DN map $\Lambda_{q,A}$ is defined as follows.

**Definition 2.** (Weak formulation of DN map)
The DN map $\Lambda_{q,A} : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)$ is defined as follows by the duality:

$$\langle \Lambda_{q,A} f, \overline{g} \rangle = \int_{\Omega} (D_A u) \cdot \overline{D_A v} + qu \overline{v} \, dx, \quad f, g \in H^{1/2}(\partial \Omega),$$

where $u \in H^1(\Omega)$ is the weak solution of (1.1) and $v \in H^1(\Omega)$ is any function satisfying $v|_{\partial \Omega} = g$.

We define the weak solution of the magnetic Schrödinger equation with a defect $D$ in $\Omega$ under the Robin boundary condition on $\partial D$.

**Definition 3.** (Robin case)
Suppose $q \in L^\infty(\Omega \setminus \overline{D}), q \geq 0, \lambda \in C^1(\partial D), \lambda \geq 0$ and $A \in C^1(\Omega \setminus \overline{D}, \mathbb{R}^n)$. Let $\nu$ is the outward unit normal vector to $\Omega \setminus \overline{D}$. For a given $f \in H^{1/2}(\partial \Omega)$, we say $u \in H^1(\Omega \setminus \overline{D})$ is a weak solution to the following value problem for the magnetic Schrödinger equation

$$\begin{cases} D_A^2 u + qu = 0 \text{ in } \Omega \setminus \overline{D}, \\ \nu \cdot (\nabla + iA) u + \lambda u = 0 \text{ on } \partial D, \\ u = f \text{ on } \partial \Omega, \end{cases} \quad (1.3)$$

if $u = f$ on $\partial \Omega$ and $u$ satisfies

$$\int_{\Omega \setminus \overline{D}} (D_A u) \cdot \overline{D_A \varphi} + qu \overline{\varphi} \, dx + \int_{\partial D} \lambda u \overline{\varphi} \, dS = 0$$

for any $\varphi \in H^1(\Omega \setminus \overline{D})$ such that $\varphi|_{\partial \Omega} = 0$.

The DN map $\Lambda_{q,A,D}^{(R)}$ is defined as follows.
Definition 4. (DN map of the Robin case)

The DN map \( \Lambda_{q,A,D}^{(R)} : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) is defined as follows by the duality:

\[
\langle (\Lambda_{q,A,D}^{(R)} f), \varphi \rangle = \int_{\partial \Omega} \lambda u \varphi dS + \int_{\Omega \setminus \overline{D}} (D_A u) \cdot D_A \overline{v} + q u \overline{v} \, dx, \quad f, g \in H^{1/2}(\partial \Omega),
\]

where \( u \in H^1(\Omega \setminus \overline{D}) \) is the weak solution of (1.3) and \( \varphi \in H^1(\Omega \setminus \overline{D}) \) is any function \( \varphi|_{\partial \Omega} = g \).

In the special case \( \lambda = 0 \), we denote \( \Lambda_{q,A,D}^{(N)} \) instead of \( \Lambda_{q,A,D}^{(R)} \).

Remark 1. The weak solution of the magnetic Schrödinger equation with a defect \( D \) in \( \Omega \) under the Dirichlet boundary condition on \( \partial D \) and the DN map \( \Lambda_{q,A,D}^{(D)} \) can be defined in a similar way.

Next, we introduce an indicator function that plays an important role in the enclosure method. We denote by \( A \) obtained from the indicator function \( I \).

Since the indicator function \( I \) is Lipschitz continuous. Let \( A \) be defined as follows:

\[
h_D(\omega) = \sup_{x \in D} x \cdot \omega, \quad \omega \in S^{n-1}.
\]

Then it is well-known that the convex hull \( \text{conv}(D) \) of \( D \) is obtained as follows.

\[
\text{conv}(D) := \cap_{\omega \in S^{n-1}} \{ x \in R^n | x \cdot \omega < h_D(\omega) \}.
\]

Since the indicator function \( I_{\omega}(\tau; t) \) is determined from the DN map, if the support function \( h_D(\omega) \) is obtained from the indicator function \( I_{\omega}(\tau; t) \), the convex hull \( \text{conv}(D) \) of inclusion \( D \) can be reconstructed from the observation data on boundary \( \partial \Omega \). Now, we give the formula of the reconstruction of the support function under a certain smallness condition for the vector potential \( A \).

Theorem 1. Suppose \( \partial D \) is Lipschitz continuous. Let \( n = 2, 3, q \in H^2(\Omega), q \geq 0, A \in H^3(\Omega) \) and \( C(\Omega)||A||_{H^3(\Omega)} \leq \frac{1}{2} \). Then, we have

\[
\lim_{\tau \to \infty} \frac{\log |I_{\omega}^{(D)}(\tau; 0)|}{2\tau} = h_D(w), \quad \lim_{\tau \to \infty} \frac{\log |I_{\omega}^{(N)}(\tau; 0)|}{2\tau} = h_D(w),
\]

for any \( \omega \in S^{n-1} \). Here, the constant \( C(\Omega) \) depends only on \( \Omega \).

For a given \( \omega \in S^{n-1} \), we further assume the following condition \( (D)_\omega \) for the Robin case.

(D)_\omega: Suppose \( \partial D \) is \( C^2 \) and the set \( T(\omega) := \{ x \in \overline{D} | h_D(\omega) - x \cdot \omega = 0 \} \) consists of only one point \( x_0 \in \partial D \). Furthermore, we assume that in the neighborhood of \( x_0 \) the boundary \( \partial D \) can be expressed as \( y = f(s), |s| < \epsilon, s \in R^{n-1} \), and there exists \( K_0, K_1 > 0, m_\omega \geq 2 \) such that

\[
K_0 |s|^{m_\omega} \leq f(s) \leq K_1 |s|^{m_\omega}, \quad (|s| < \epsilon).
\]

Theorem 2. (Robin case) Suppose \( \lambda \neq 0, \lambda \geq 0 \) and \( \lambda \in C^1(\partial D) \). Let \( n = 2, 3, q \in H^2(\Omega), q \geq 0, A \in H^3(\Omega) \) and \( C(\Omega)||A||_{H^3(\Omega)} \leq \frac{1}{2} \). We assume that the condition \( (D)_\omega \) holds as \( 2 \leq m_\omega < 3 \) for some \( \omega \in S^{n-1} \). Then, we have

\[
\lim_{\tau \to \infty} \frac{\log |I_{\omega}^{(R)}(\tau; 0)|}{2\tau} = h_D(\omega).
\]

References
